MT240: Complex Variables and Transforms

Definitions and Theorems

Spring 2019

1 Complex Variable

1.1 Roots of a polynomial

A root or zero of a polynomial $P(z)$ is a number $z_i \in \mathbb{C}$ for which $P(z_i) = 0$.

Theorem 1.1 (Fundamental Theorem of Algebra)**.** *A polynomial $P(z)$ of degree n has n roots, some of them possibly degenerate (repeated).*

Theorem 1.2 (Conjugate Root Theorem)**.** *If a polynomial $P(z)$ with real coefficients has a root $z_o = a + bi$ (with $a, b \in \mathbb{R}$, $b \neq 0$) then $\overline{z}_o = a - bi$ is also a root of $P(z)$.*

1.2 Circle of radius r, centered at z_o

Set of all points z at a distance r from z_o.

$$K_r(z_o) = \{z \in \mathbb{C} : |z - z_o| = r\}$$

1.3 Open disk of radius r, centered at z_o

Set of all points z at a distance less than r from z_o.

$$B_r(z_o) = \{z \in \mathbb{C} : |z - z_o| < r\}$$

1.4 Bounded set

A set $E \subset \mathbb{C}$ is bounded if there exists a number $R > 0$ such that $E \subset B_R(0)$. If no such R exists then E is called unbounded.

1.5 Interior point

A point z is an interior point of a set $E \subset \mathbb{C}$ if there exists some $r > 0$ such that $B_r(z) \subset E$.

The interior of E, represented as $\mathrm{int}(E)$, is the set of all interior points of E.

1.6 Boundary point

A point z is a boundary point of E if every $B_r(z)$ for $r > 0$ contains a point in E and a point not in E.

The boundary of the set E, represented as ∂E, is the set of all boundary points of E.

The closure of E, represented as $\mathrm{cl}(E)$, is the set E together with all of its boundary points, i.e., $\mathrm{cl}(E) = E \cup \partial E$.

1.7 Open set

A set E is open if all of its points are its interior points.

1.8 Closed set

A set E is closed if it contains all of its boundary points.

1.9 Theorem about connectedness of open sets

An open set E is connected if and only if any two points in E can be joined in E by successive line segments.

1.10 Extended Complex Plane

The extended complex plane $\hat{\mathbb{C}}$ consists of the entire complex plane $\mathbb{C}$ together with the point at infinity, i.e. $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

2 Complex Functions

A complex function $f : \mathbb{C} \to \mathbb{C}$ is a rule that assigns to each element in its domain D one and only one element in a set E, the range of f.

For the complex variable $z = x + iy$,

$$w = f(z) = u(x, y) + iv(x, y).$$

2.1 Limit of a function

Let f be a function defined in some neighborhood of $z_o \in \mathbb{C}$, except possibly at z_o itself.

Limit of $f(z)$ as $z \to z_o$ is the number w_o

$$\lim_{z \to z_o} f(z) = w_o$$

if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - w_o| < \varepsilon$ whenever $0 < |z - z_o| < \delta$.

2.2 Continuous function

A function f is continuous at a point z_o in its domain if the limit

$$\lim_{z \to z_o} f(z) = f(z_o).$$

2.3 Differentiable function

A function f is complex differentiable at $z_o \in \mathbb{C}$ if the limit

$$\lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_o}$$

exists. This limit is often referred to as $f'(z_o)$ i.e. the derivative of f at z_o.

Another way to represent this limit is

$$\lim_{\Delta z \to 0} \frac{f(z_o + \Delta z) - f(z_o)}{\Delta z},$$

where $\Delta z = z - z_o$.

This is the fundamental definition of a derivative and is called 'derivative from first principles'.

Theorem 2.1 (Differentiability implies continuity)**.** *If a function f is differentiable at z_o, then it is continuous at z_o.*

This theorem also implies that if f is discontinuous at a point z_o, then is not differentiable at this point. However, this does not mean that a function continuous at a point z_o is necessarily differentiable at z_o.

2.4 Partial Derivative

A partial derivative is defined as derivative of a function $f(x_1, x_2, ..., x_n)$ of multiple variables when all the variables, except the variable of interest, are held constant during the differentiation.

Partial derivative of f with respect to variable x_i is defined as

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, ..., x_i + \Delta x_i, ..., x_n) - f(x_1, ..., x_i, ..., x_n)}{\Delta x_i}$$

We will use the short-hand notation f_x to represent the partial derivative $\dfrac{\partial f}{\partial x}$.

2.5 Analytic function

A function f is said to be analytic (or holomorphic) on an open set $E \subset \mathbb{C}$ when it is differentiable at each point $z \in E$.

Theorem 2.2. *If $f(z) = u(x, y) + iv(x, y)$ is analytic at $z_o \in \mathbb{C}$, then the partial derivatives u_x, v_x, u_y and v_y exist at z_o and satisfy*

$$u_x = v_y, \tag{2.1}$$
$$u_y = -v_x. \tag{2.2}$$

These are known as Cauchy-Riemann equations and can also be represented as

$$f_x = -if_y. \tag{2.3}$$

Also, if f is analytic on domain D then its derivative

$$f'(z) = f_x = -if_y. \tag{2.4}$$

Theorem 2.3. *Let $f(z) = u(x, y) + iv(x, y)$ be defined on a domain $D \in \mathbb{C}$. The function f is analytic on D if and only if its first partial derivatives u_x, v_x, u_y and v_y are continuous on D and satisfy Cauchy-Riemann equations.*

Theorem 2.4. *If f is analytic on a domain D, and if $f'(z) = 0$ for all $z \in D$, then f is constant in D.*

2.6 Polar Form of Cauchy-Riemann Equations

If $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic at $z_o \in \mathbb{C}$, then the partial derivatives u_r, u_θ, v_r and v_θ exist at z_o and satisfy

$$ru_r = v_\theta, \tag{2.5}$$
$$u_\theta = -rv_r. \tag{2.6}$$

Also, if f is analytic on the domain D then its derivative

$$f'(z) = e^{-i\theta} f_r = -i\frac{e^{-i\theta}}{r} f_\theta. \tag{2.7}$$

2.7 Singular point

If f is not analytic at a point $z_o \in \mathbb{C}$ but analytic at some point in every neighborhood of z_o, then z_o is called a singular point (or a singularity) of f.

2.8 Harmonic functions

A real-valued function $f(x, y)$ is said to be harmonic in a domain D if all its second-order partial derivatives f_{xx}, f_{yy} and f_{xy} are continuous in D and if, at each point of D, f satisfies Laplace equation,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

In polar coordinates, for a harmonic function $f(r, \theta)$, the Laplace equation can be written as

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0.$$

Theorem 2.5. *If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then each of the functions $u(x, y)$ and $v(x, y)$ is harmonic in D.*

2.9 Elementary Functions

2.9.1 Rational Function

$$R(z) = \frac{b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m}{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n}$$

Roots of the numerator polynomial are called *zeros* and the roots of the denominator polynomial are called *poles* of the function. If the order of the numerator polynomial is m and the order of the denominator polynomial is n, there are m zeros and n poles of the rational function. In terms of zeros and poles, the rational function can be represented in the factorized form as follows,

$$R(z) = k\frac{(z - z_1)(z - z_2)\cdots(z - z_m)}{(z - p_1)(z - p_2)\cdots(z - p_n)},$$

where $k = \dfrac{b_m}{a_n}$.

2.9.2 Exponential Function

$$f(z) = e^z \quad \text{for } z \in \mathbb{C}$$

2.9.3 Complex Trigonometric Functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{for } z \in \mathbb{C}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{for } z \in \mathbb{C}$$

2.9.4 Logarithm Function

The principal branch of the logarithm function is the inverse of the exponential function $f(z) = e^z$ for $\{z \in \mathbb{C} : -\pi < \text{Im } z \leq \pi\}$. It is defined as follows.

$$\text{Log } z = \ln |z| + i\text{Arg } z \quad \text{for } \{z \in \mathbb{C} : z \neq 0\}$$

The function is not analytic at the origin and the negative real-axis, i.e., $\{z \in \mathbb{C} : \text{Re } z \leq 0 \wedge \text{Im } z = 0\}$.

2.10 The Inverse Function

Given a function $f(z)$ that maps domain D to R. Its inverse function $f^{-1}(z)$ is defined by

$$f^{-1}(f(z)) = f(f^{-1}(z)) = z,$$

and maps R to D.

Theorem 2.6. *A function $f(z)$ has an inverse $f^{-1}(z)$ if and only if it is bijective (one-to-one and onto).*

However, inverse functions are commonly defined for elementary functions that are multivalued in the complex plane. In such cases, the inverse relation holds on some subset of the complex plane. But, over the whole plane, either or both parts of the identity $f^{-1}(f(z)) = f(f^{-1}(z)) = z$ may fail to hold. For example, consider the function $\log z$.

Theorem 2.7. *If $f(z)$ is analytic at z_o and $f'(z_o) \neq 0$, then there is an open disk D centered at z_o such that $f(z)$ is one-to-one on D.*

Theorem 2.8 (Inverse function theorem)**.** *Suppose that $f(z)$ is a function analytic one some domain D and maps it to R, and there exists a continuous function $g(z)$ with domain R such that $g(f(z)) = z$ for all $z \in D$ (which simply means $g = f^{-1}$). Then g is analytic in R, and*

$$g'(z) = \frac{1}{f'(g(z))} \quad \textit{for } z \in R.$$

Theorem 2.9 (Inverse function theorem)**.** *If $f(z)$ is analytic at z_o and $f'(z_o) \neq 0$, then $f^{-1}(w)$ is analytic at w_o and*

$$\frac{df^{-1}}{dw}(w_o) = \frac{1}{\dfrac{df}{dz}(z_o)}$$

where $w_o = f(z_o)$.

2.11 Möbius transformation

A Möbius transformation is a function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$.

Also known as fractional linear transformation or bilinear transformation, it is a bijective map in the extended complex plane ($\hat{\mathbb{C}}$). Every Möbius transformation maps circles and lines to circles or lines. The composition of two Möbius transformations is a Möbius transformation. The inverse of a Möbius transformation is also a Möbius transformation.

Theorem 2.10. *Given three distinct points $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and three distinct points $w_1, w_2, w_3 \in \hat{\mathbb{C}}$, there exists a unique Möbius transformation f such that $f(z_1) = w_1$, $f(z_2) = w_2$, and $f(z_3) = w_3$.*

Every Möbius transformation is the composition of maps of the type

$$\begin{aligned} &z \mapsto az \quad \text{rotation and dilation (scaling)} \\ &z \mapsto z + b \quad \text{translation} \\ &z \mapsto \frac{1}{z} \quad \text{inversion} \end{aligned}$$

The linear transformation $z \mapsto az + b$, which is a composition of rotation, dilation and translation, is called Affine transformation. So a Möbius transformation is a composition of Affine transformation and inversion.

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3 Complex Integration

If $\gamma : [a, b] \to \mathbb{C}$ is a smooth curve and f is continuous on γ, then

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Theorem 3.1 (Independence of parameterization)**.** *Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve, and let $\beta : [c, d] \to \mathbb{C}$ be another smooth parameterization of the same curve, given by $\beta(s) = \gamma(h(s))$, where $h : [c, d] \to [a, b]$ is a smooth bijection. Let f be a complex-valued function, defined on γ. Then*

$$\int_\beta f(z)dz = \int_\gamma f(z)dz$$

3.1 Primitive

Let $D \subset \mathbb{C}$ be a domain, and let $f : D \to \mathbb{C}$ be a continuous function. A primitive of f on D is an analytic function $F : D \to \mathbb{C}$ such that $F' = f$ on D.

Theorem 3.2 (Fundamental theorem of calculus for complex analytic functions)**.** *If f is continuous on a domain D and if f has a primitive F in D, then for any curve $\gamma : [a, b] \to D$ we have*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

Theorem 3.3 (Goursat's theorem)**.** *Let D be a simply connected domain in $\mathbb{C}$, and let f be analytic in D. Then f has a primitive in D. Moreover, a primitive is given explicitly by picking $z_o \in D$ and letting*

$$F(z) = \int_{z_o}^{z} f(\hat{z})d\hat{z}$$

where the integral is taken on an arbitrary curve in D from z_o to z.

3.2 Length of a curve

Length of a curve $\gamma : [a, b] \to \mathbb{C}$.

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)|dt$$

3.3 Cauchy's integral theorem

Theorem 3.4 (Cauchy's integral theorem)**.** *Let D be a simply connected domain in $\mathbb{C}$, and let f be analytic in D. Let $\gamma : [a, b] \to D$ be a piecewise smooth and closed curve in D (i.e. $\gamma(b) = \gamma(a)$). Then*

$$\oint_\gamma f(z)dz = 0.$$

Corollary 3.4.1. *Let γ_1 and γ_2 be two simple closed curves (i.e. neither of the curves intersects itself), oriented counterclockwise, where γ_2 is inside γ_1. If f is analytic in a domain D that contains both curves as well as the region between them, then*

$$\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz$$

Theorem 3.5 (Cauchy's Integral Formula)**.** *Let D be a simply connected domain, bounded by a piecewise smooth curve γ, and let f be analytic in a set U that contains the closure of D (i.e. D and γ). Then*

$$\oint_\gamma \frac{f(z)}{z-z_o}dz = 2\pi i f(z_o) \quad \text{for all } z_o \in D.$$

For higher powers of $(z - z_o)$, we have Cauchy's Integral formula for derivatives as follows,

$$\oint_\gamma \frac{f(z)}{(z-z_o)^{k+1}}dz = \frac{2\pi i}{k!} f^{(k)}(z_o) \quad \text{for all } z_o \in D.$$

Here is another amazing theorem which follows from Cauchy's Integral formula.

Theorem 3.6. *If f is analytic in an open set U, then f' is also analytic in U.*

3.4 Isolated singularities

A point z_o is an isolated singularity of f is f is analytic in a punctured disk $\{0 < |z - z_o| < r\}$ centered at z_o.

3.4.1 Types of isolated singularities

Suppose z_o is an isolated singularity of an analytic function f. Then the singularity z_o is

- *removable* if $\lim_{z\to z_0} (z - z_0)f(z) = 0$
- a *pole* of order N if $\lim_{z\to z_0} (z - z_0)^N f(z) = c$, where $N \in \mathbb{Z}^+$ and $c \in \mathbb{C}$ is a non-zero constant.
- *essential* (out of the scope of this course).

Theorem 3.7 (Residue theorem)**.** *Let D be a simply connected domain, and let f be analytic in D, except for isolated singularities. Let γ be a simple closed curve in D (oriented counterclockwise), and let $z_1, z_2, \ldots, z_n$ be isolated singularities of f that lie inside of γ. Then*

$$\oint_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k).$$

If z_k is a removable singularity or a simple pole (order $N = 1$), then

$$\mathrm{Res}(f, z_k) = \lim_{z\to z_k} (z - z_k)f(z),$$

and if z_k is a pole of order $N > 1$, then

$$\mathrm{Res}(f, z_k) = \frac{1}{(N-1)!} \lim_{z\to z_k} \frac{d^{N-1}}{dz^{N-1}} (z - z_k)^N f(z)$$

4 Transforms

4.1 Complex Fourier Series

Let $f(t)$ be a periodic function with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$. Then the complex Fourier coefficients c_n of $f(t)$, whenever they exist, are defined by

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-in\omega_0 t}dt \quad \text{for } n \in \mathbb{Z}.$$

When c_n are the complex Fourier coefficients of the periodic function $f(t)$ with Complex Fourier series period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$, then the complex Fourier series of $f(t)$ is defined by

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t},$$

4.1.1 Trigonometric Fourier series

For a real-valued piecewise smooth periodic function $f(t)$, its Fourier series can also be represented as follows

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t,$$

where

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos n\omega_0 t \; dt, \quad b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin n\omega_0 t \; dt$$

Theorem 4.1 (Fundamental Theorem of Fourier series)**.** *Suppose that $f(t)$ is a piecewise smooth periodic function on $\mathbb{R}$ with Fourier series coefficients c_n. Then for any $t \in \mathbb{R}$*

$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} = \frac{1}{2}\left(f(t_+) + f(t_-)\right).$$

Theorem 4.2 (Uniqueness Theorem)**.** *Let $f(t)$ and $g(t)$ be piecewise smooth periodic functions with Fourier series coefficients c_n and d_n. If $c_n = d_n$ for all $n \in \mathbb{R}$ then $f(t) = g(t)$ at all points where $f(t)$ and $g(t)$ are continuous.*

4.1.2 Parseval's identity

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

where the integral $\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt$ is called the average power of $f(t)$. So the average power in a piecewise smooth periodic function $f(t)$ per time period is equal to the absolute squared sum of its Fourier series coefficients.

4.1.3 Properties of Complex Fourier Series

Property	Function	Fourier Series Coefficients
Linearity	$af(t) + bg(t)$	$ac_n + bd_n$
Conjugation	$\overline{f(t)}$	$\overline{c_{-n}}$
Time shift	$f(t - t_0)$	$e^{-in\omega_0 t_0} c_n$
Time reversal	$f(-t)$	c_{-n}

4.2 Fourier Transform

The Fourier transform is the generalization of complex Fourier series in the limit as $T \to \infty$. For a given function $f(t)$, its Fourier transform $F(\omega)$ for $\omega \in \mathbb{R}$ is defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt.$$

Once we have $F(\omega)$, we can transform back to $f(t)$ using the Fourier inversion formula, also known as the inverse Fourier transform, using the Fourier integral as follows

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega.$$

Theorem 4.3. *Suppose that $f(t)$ is an absolutely integrable function, i.e.*

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty,$$

and is piecewise smooth on every bounded interval. Then its Fourier transform $F(\omega)$ exists.

Theorem 4.4 (Fundamental theorem of Fourier integral)**.** *Let $f(t)$ be an absolutely integrable and piecewise smooth function on $\mathbb{R}$ and let $F(\omega)$ be the Fourier transform of $f(t)$. Then the Fourier integral converges for each $t \in \mathbb{R}$ as*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega = \frac{1}{2}(f(t_+) + f(t_-)).$$

Theorem 4.5 (Parseval's Theorem)**.** *Let $f(t)$ be a piecewise smooth square integrable function. Then*

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

The integral $\displaystyle\int_{-\infty}^{\infty} |f(t)|^2\, dt$ is called total energy of the function. Square integrable functions have finite energy and are called 'energy signals'.

According to the theorem, the total energy of a function $f(t)$ can be calculated by integrating either power across time or spectral power across frequency, where power p in a function f at time t is defined as $p(t) = |f(t)|^2$.

4.2.1 Properties of Fourier Transform

Property	Function	Fourier Transform
Linearity	$af(t) + bg(t)$	$aF(\omega) + bG(\omega)$
Conjugation	$\overline{f(t)}$	$\overline{F(-\omega)}$
Time shift	$f(t - t_0)$	$e^{-i\omega t_0}F(\omega)$
Frequency shift	$e^{i\omega_0 t}f(t)$	$F(\omega - \omega_0)$
Time scaling	$f(ct)$	$\frac{1}{\lvert c\rvert}F\left(\frac{\omega}{c}\right)$
Time reversal	$f(-t)$	$F(-\omega)$
Differentiation in time domain	$f^{(n)}(t)$	$(i\omega)^n F(\omega)$
Differentiation in frequency domain	$(-it)^n f(t)$	$F^{(n)}(\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Convolution in time domain	$f(t) * g(t)$	$F(\omega)G(\omega)$
Convolution in frequency domain	$f(t)g(t)$	$\frac{1}{2\pi}F(\omega) * G(\omega)$

4.2.2 Fourier Transform Table

Rectangle, triangle and pulse functions are defined in class.

	Function	Fourier Transform
Rectangle	$\sqcap_a(t)$	$\dfrac{a\sin(\frac{a\omega}{2})}{\frac{a\omega}{2}}$
Triangle	$\wedge_a(t)$	$\dfrac{a\sin^2(\frac{a\omega}{2})}{(\frac{a\omega}{2})^2}$
Pulse	$p_a(t)$	$-i\dfrac{a\sin^2(\frac{a\omega}{4})}{\frac{a\omega}{4}}$
Absolute-timed Exponential	$e^{-a\|t\|}$, $a>0$	$\dfrac{2a}{a^2+\omega^2}$
Causal Exponential	$e^{-at}u(t)$, $a>0$	$\dfrac{1}{a+i\omega}$
Gaussian	e^{-at^2}	$\sqrt{\dfrac{\pi}{a}}\ e^{-\frac{\omega^2}{4a}}$
Dirac-delta function (Unit impulse)	$\delta(t)$	1
Unit step function	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{i\omega}$
Sign function	$\mathrm{sgn}(t)$	$\dfrac{2}{i\omega}$
Complex exponential	$e^{i\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
Cosine	$\cos\omega_0 t$	$\pi(\delta(\omega+\omega_0)+\delta(\omega-\omega_0))$
Sine	$\sin\omega_0 t$	$i\pi(\delta(\omega+\omega_0)-\delta(\omega-\omega_0))$
Impulse train	$\displaystyle\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\displaystyle\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi}{T}k\right)$

4.3 Laplace Transform

For a given function $f(t)$, its Laplace transform $F(s)$ for $s \in \mathbb{C}$ is defined as

$$F(s) = \int_{0_-}^{\infty} f(t)e^{-st}dt,$$

where variable $s = \sigma + i\omega$ for $\sigma, \omega \in \mathbb{R}$.

4.3.1 Absolute Convergence

In general, the absolute convergence of an indefinite integral also implies its own converges. So if the Laplace transform integral of a function $f(t)$ converges absolutely, then the integral itself converges and the Laplace transform of $f(t)$ exists.

For a given causal function $f(t)$, there exists a number $\sigma \in \mathbb{R}$ with $-\infty \leq \sigma \leq \infty$ such that the Laplace transform integral

$$\int_{0}^{\infty} f(t)e^{-st}dt$$

is absolutely convergent i.e.

$$0 \leq \left| \int_{0}^{\infty} f(t)e^{-st}dt \right| < \infty$$

for all $s \in \mathbb{C}$ with Re $s > \sigma$ and not absolutely convergent for all $s \in \mathbb{C}$ with Re $s < \sigma$. The region Re $s > \sigma_a$ is called the region of absolute convergence of $f(t)$, and the value Re $s = \sigma$ is called abscissa of absolute convergence.

It is important to note that the absolute integral reduces to

$$\left| \int_{0}^{\infty} f(t)e^{-st}dt \right| = \int_{0}^{\infty} |f(t)|\, e^{-\sigma t}dt$$

4.3.2 Exponential Order Functions

The functions that do not grow faster than the exponential function $e^{\sigma t}$ for $t \geq 0$ are called exponential order functions. The Laplace transform of an exponential order function exists because the Laplace transform integral absolutely converges for such functions.

There are two conditions to check if a function is of exponential order.

(1) $|f(t)|e^{-st} < M$ for some $M > 0$

(2) $\lim_{t \to \infty} f(t)e^{-st} = 0$

4.3.3 Properties of Laplace Transform

Property	Function	Laplace Transform
Linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
Time shift	$f(t - t_0)$	$e^{-t_0 s}F(\omega)$
s-shift	$e^{s_0 t}f(t)$	$F(s - s_0)$
Scaling	$f(ct), c > 0$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
Convolution in time domain	$f(t) * g(t)$	$F(s)G(s)$

4.3.4 Table of Laplace Transform

	Function	Laplace Transform
Dirac-delta	$\delta(t)$	1
Unit step	$u(t)$	$\frac{1}{s}$
Exponential	e^{at}	$\dfrac{1}{s-a}$
Complex exponential	e^{ibt}	$\dfrac{1}{s-ib}$
Cosine	$\cos bt$	$\dfrac{s}{s^2+b^2}$
Sine	$\sin bt$	$\dfrac{b}{s^2+b^2}$
Polynomial	t^n	$\dfrac{n!}{s^{n+1}}$
First time-derivative	$f'(t)$	$sF(s) - f(0_-)$
Second time-derivative	$f''(t)$	$s^2F(s) - sf(0_-) - f'(0_-)$
n^{th} time-derivative	$f^{(n)}(t)$	$s^nF(s) - \sum_{k=0}^{n-1} s^{n-k-1}f^{(k)}(0_-)$
Time integration	$\int_{0_-}^{t} f(\tau)d\tau$	$\dfrac{F(s)}{s}$

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