MT240: Complex Variables and Transforms

Midterm Exam (Spring 2019)

Instructor: Usama Bin Sikandar **March 25, 2019**

Student Name: ____________________

Roll Number: ____________________

100 Marks

120 minutes

Instructions:

- There are **13** printed pages and **5** blank page.
- **All problems** are compulsory.
- **Calculators** are strictly **NOT** allowed.
- Write **all your work in this booklet**, including any rough work.
- Read the statement **carefully** before you start attempting a problem.
- Properly **label all the axes and relevant points** if you draw any graphs.
- You are allowed to get help **from a hard copy of lecture notes** uploaded on Google Classroom.

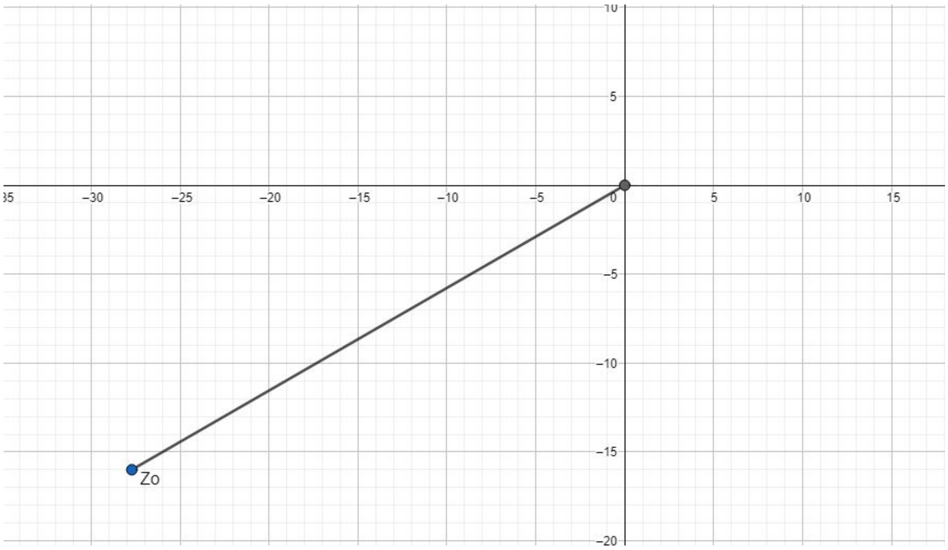
Blank page for marks and contestation.
Do NOT write anything on this page.

P1	**P2**	**P3**	**P4**	**P5**	**P6**	**Total**
15	**20**	**15**	**15**	**20**	**15**	**100**

Problem 1 [15 marks]

Consider the complex number $z_0 = -16\sqrt{3} - 16i$.

(a) Plot z_0 in the complex plane.



(b) Find $|z_0|$ and Arg $|z_0|$.

Solution

$|Z_o| = 16\sqrt{3+1} = 32$

$Arg|z_o| = -\pi + tan^{-1}(\frac{1}{\sqrt{3}}) = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$

(c) Find all the values of z for which $z^5 = z_0$

Solution

$Z^5 = 32e^{-i\frac{5\pi}{6}} = 32e^{-i\frac{5\pi}{6}+i2n\pi}$

$z = 2e^{-i\frac{\pi}{6}+\frac{2n\pi}{5}}$,where n = -2,-1,-,1,2

$z_1 = 2e^{-i\frac{29\pi}{30}}$

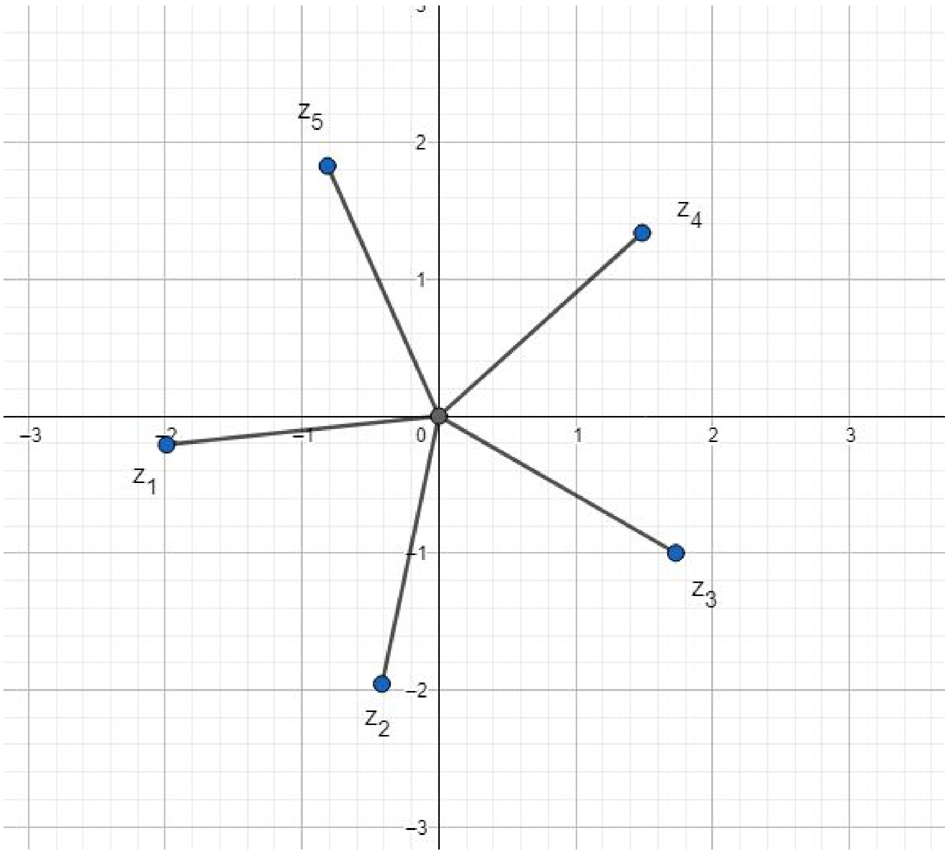
$z_2 = 2e^{-i\frac{17\pi}{30}}$

$z_3 = 2e^{-i\frac{5\pi}{30}}$

$z_4 = 2e^{i\frac{7\pi}{30}}$

$z_5 = 2e^{i\frac{19\pi}{30}}$

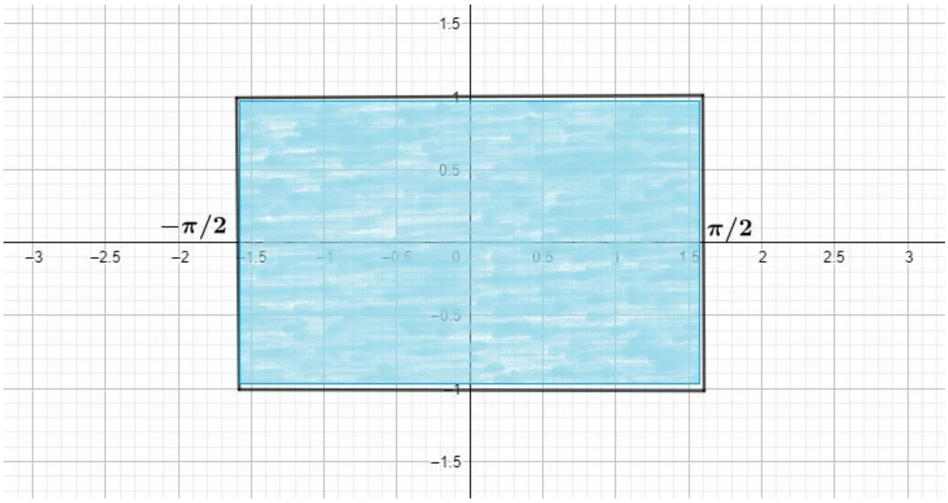
(d) On the following complex plane, plot the values of z found in part (c).



Problem 2 [20 marks]

Consider the set
$$D = \left\{ z \in \mathbb{C} : -\frac{\pi}{2} < \text{Re } z < \frac{\pi}{2} \quad \wedge \quad -1 < \text{Im } z < 1 \right\}.$$

(a) Sketch D in the complex plane.



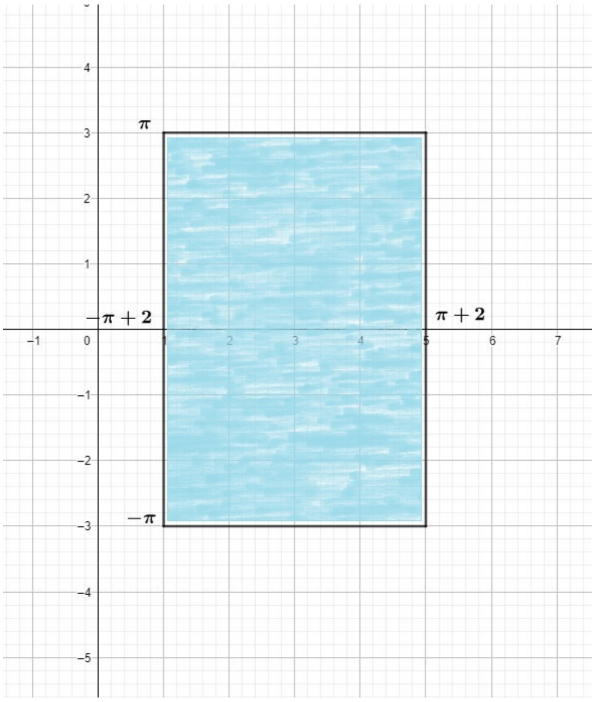
(b) Show that D is open and connected.

Solution:

No boundary pints / All interior points

Any two points in D can be joined by a line segment in D

(c) The function $f(z) = 2iz + \pi$ maps the set D to a set E. Sketch E in the complex plane.



(d) Which linear transformations are caused by $f(z)$?

Rotation, Scaling and translation

(e) Write the set E in set-builder notation.

$$E = \{w \in \mathbb{C} : \pi - 2 < \text{Re } z < \pi + 2 \quad \wedge \quad -\pi < \text{Im } z < \pi\}.$$

(f) Find a function that maps E back to D.

$f^{-1}(w) = z$

$w = f(z) = 2iz + \pi$

$z = \dfrac{w - \pi}{2i}$

$z = \dfrac{i}{2}(\pi - w)$

Problem 3 [15 marks]

Consider the function $f(z) = \overline{e^z}$.

(a) Find a set $D \in \mathbb{C}$ on which $f(z)$ is continuous. [Hint: $f(z) = u(x, y) + iv(x, y)$.]

$f(z) = e^x cosy - ie^x siny$

The above function will be continuous at $z_o = x_o + iy_o$

$$\lim_{z \to z_o} f(z) = f(z_o)$$

f(z) will be continuous if U(x,y) and V(x,y) are continuous

(1) $\underline{U(x, y)}$

$$\lim_{x \to x_o, y \to y_o} e^x cosy = e^x cosy, [Cont.\forall(z \in \mathbb{C})]$$

(2) $\underline{U(x, y)}$

$$\lim_{x \to x_o, y \to y_o} -e^x siny = -e^x siny, [Cont.\forall(z \in \mathbb{C})]$$

As U(x,y) and V(x,y) are continuous and f(z) is also continuous $\forall(z \in \mathbb{C})$

(b) Find a set $E \in \mathbb{C}$ on which $f(z)$ is analytic.
Solution Analytic conditions:

(1) U_x, U_y, V_x, V_y continuous

(2) Satisfy CR

$U_x = e^x coy$ $\quad$ $V_x = -e^x sin$

$U_y = -e^x sin$ $\quad$ $V_y = -e^x coy$

(1) Partial Derivatives are Continuous

(2) $U_x = V_y$
$2e^x cosy = 0$
$e^x = 0[not - possible], cosy = 0$

$U_y = V_x$
$2e^x siny = 0$
$e^x = 0[not - possible], siny = 0$

$cosy = siny = 0$ **not possible for any value of y**
Not analytic anywhere

Problem 4 [15 marks]

Consider the Möbius transformation $f(z) = \dfrac{z-3}{z+3}$ for $z \in \hat{\mathbb{C}}$ (extended complex plane).

Convert each of your answer in the form $x + iy$.

(a) Find the value of z for which

(i) $f(z) = 0$
$z = 3$

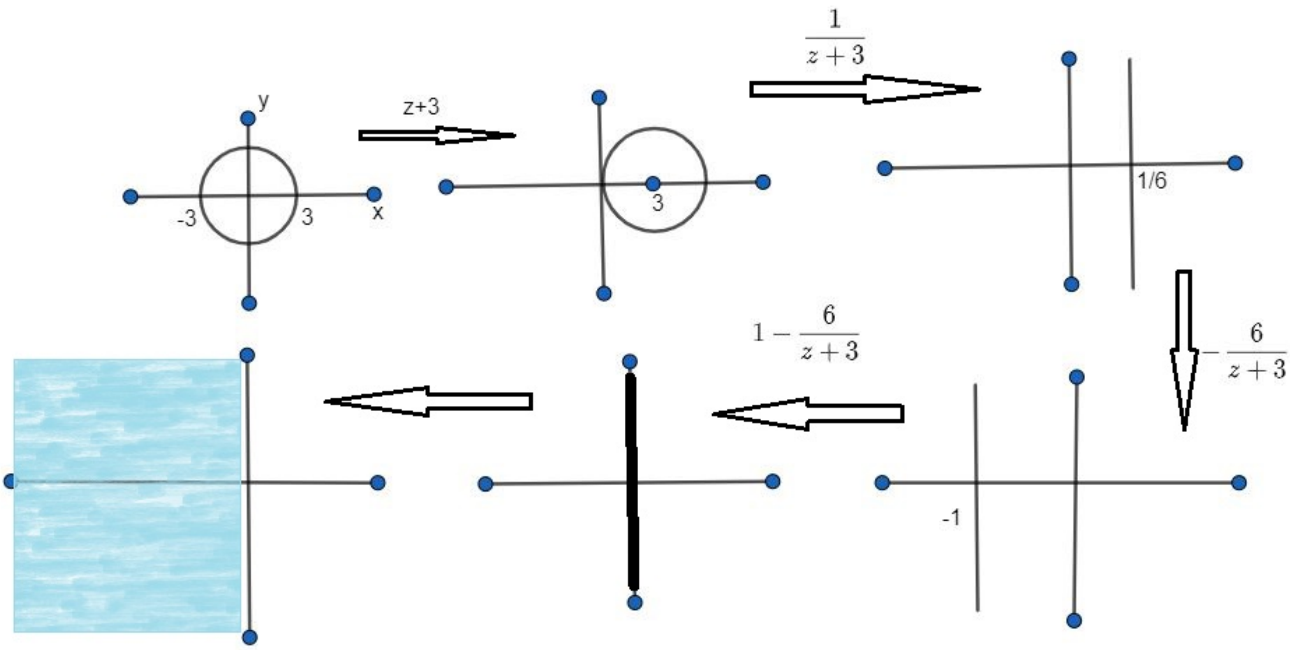
(ii) $f(z) = 1$
$z = \infty$

(iii) $f(z) = i$
$z = 3i$

(iv) $f(z) = \infty$
$z = -3$

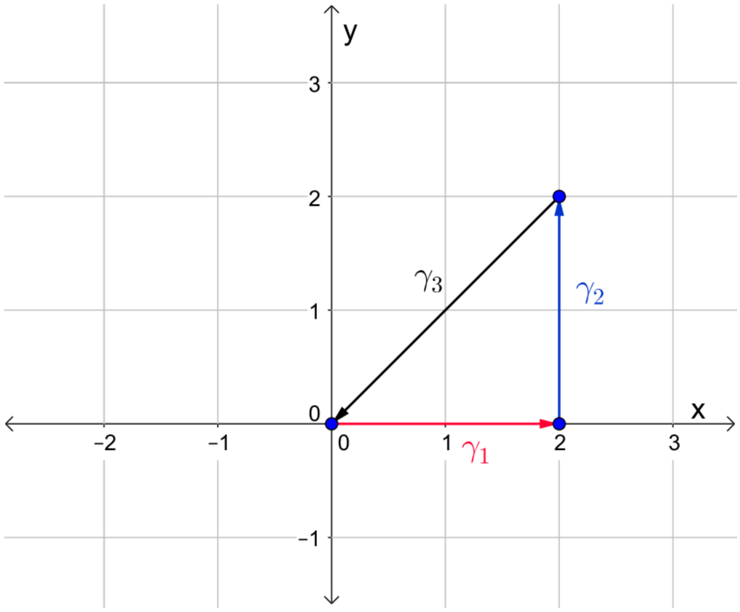
(b) Under this Möbius transformation, find and sketch the image of open disk of radius 3 centered at origin. Note: The image will be a region and not just a line or a curve.

$f(z) = 1 - \dfrac{6}{z+3}$



Problem 5 [20 marks]

It would be a better idea to first solve Problem 3 before attempting this problem. Consider a closed contour γ in the complex plane, which starts and ends at origin as shown in the figure below. Let's break it down into piecewise smooth paths γ_1, γ_2 and γ_3 marked in the figure.



(a) Parameterize each of these three paths using an appropriate interval for the parameter t in each case.
Solution

(a) $\underline{y = 0} \longrightarrow \gamma_1(t) = x + iy = t$, Where ($t = 0$ to $t = 2$)

(b) $\underline{x = 2} \longrightarrow \gamma_2(t) = 2 + it$, Where ($t = 0$ to $t = 2$)

(c) $\underline{y = x} \longrightarrow \gamma_3(t) = t + it$, Where ($t = 2$ to $t = 0$)

$\int_\gamma \overline{e^z} dz = \int_\gamma e^{x-iy} dz = \int_a^b (f(\gamma_1(t)))(\gamma'(t))\ dt$

(b) Evaluate $\int_{\gamma_1} \overline{e^z} dz$.

$f(\gamma_1(t)) = e^t$
$\gamma'(t) = 1$
$\implies \int_{\gamma_1} \overline{e^z} dz = \int_0^2 e^t\ dt = e^2 - 1$

(c) Evaluate $\int_{\gamma_2} \overline{e^z} dz$.

$f(\gamma_1(t)) = e^{2-it}$
$\gamma'(t) = i$
$\implies \int_{\gamma_2} \overline{e^z} dz = \int_0^2 ie^{2-it}\ dt = e^2(1 - e^{-2i})$

(d) Evaluate $\int_{\gamma_3} \overline{e^z} dz$.

$f(\gamma_1(t)) = e^{t-it}$
$\gamma'(t) = 1 + i$
$\implies \int_{\gamma_3} \overline{e^z} dz = \int_2^0 (1-i)e^{t-it}\ dt = i(1 - e^{2-2i})$

(e) Evaluate $\oint_\gamma \overline{e^z} dz$, using your answers to (b), (c) and (d).

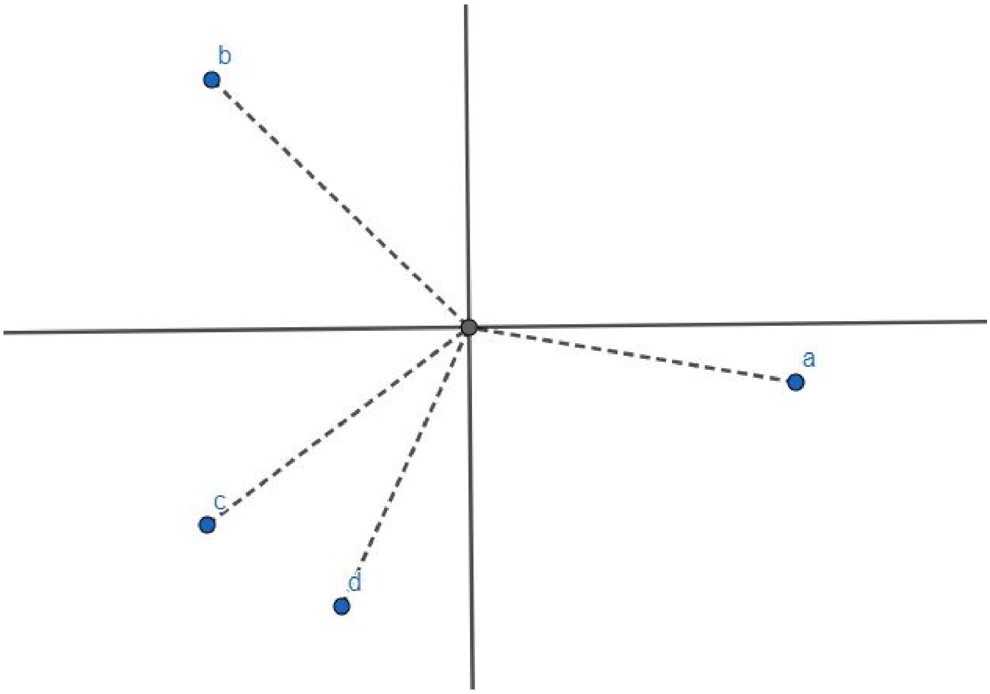
$\oint_\gamma \overline{e^z} dz = e^2 - 1 + e^2(1 - e^{-2i}) + i(1 - e^{2-2i})$= i+$e^2 - e^{-2i} - e^2 e^{-2i}$

(f) Using your answer to (e), explain whether the function $\overline{e^z}$ is analytic in the region bounded by γ?

$\oint_\gamma \overline{e^z} dz \neq 0$ So, given function is not analytic in the region bounded by γ

Problem 6 [15 marks]

Evaluate the following, giving your answer in the form $x + iy$. Then roughly plot each of these points on the following graph and mark them with (a), (b), (c) and (d). This means you will get the full credit even if you plot in the correct quadrants.



(a) $\text{Log}(-16\sqrt{3} - 16i)$

$= ln32 - i5\frac{\pi}{6}$
$= 5ln2 - i5\frac{\pi}{6}$

(b) $(-16\sqrt{3} - 16i)^{-i}$

$= e^{-iLog(-16\sqrt{3}-16i)}$
$= e^{-iln32-\frac{5\pi}{6}}$
$= e^{-\frac{5\pi}{6}}cos(ln32) - ie^{-\frac{5\pi}{6}}sin(ln32)$

(c) $e^{i(-16\sqrt{3}-16i)}$

$= e^{16}cos(16\sqrt{3}) - ie^{16}sin(16\sqrt{3})$

(d) $\cos(-16\sqrt{3} - 16i)$

$= \dfrac{e^{iz} + e^{-iz}}{z}$

$= \dfrac{e^{16}cos(16\sqrt{3}) - ie^{16}sin(16\sqrt{3}) + e^{-16}cos(16\sqrt{3}) + e^{-16}sin(16\sqrt{3})}{2}$

$= \dfrac{(e^{16} + e^{-16})cos(16\sqrt{3}}{2} - i\dfrac{(e^{16} + e^{-16})sin(16\sqrt{3}}{2}$

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