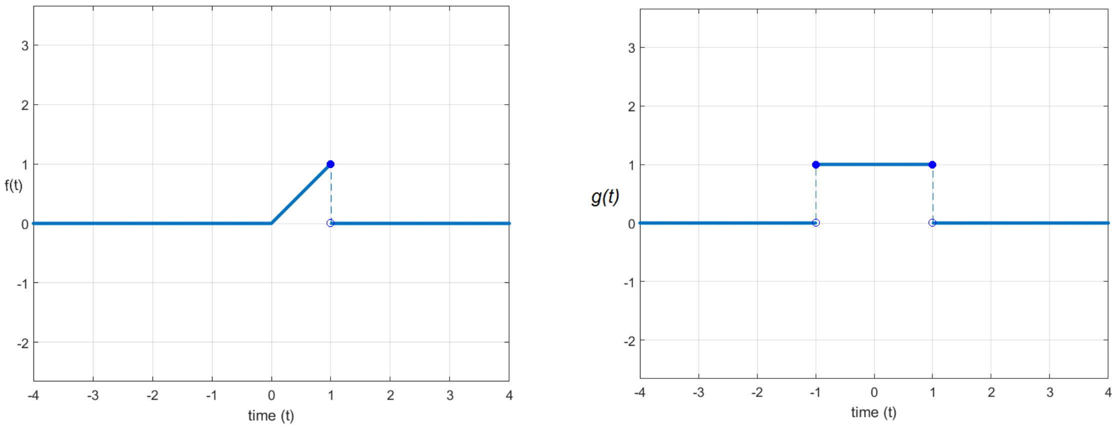
MT240: Complex Variables and Transforms

Homework 6 Solution

Spring 2018

Problem 1

In each case, evaluate the convolution product $f(t)*g(t)$. Write your answer in terms of t for appropriate range of values of t.



(a)

Solution

for: $-1 < t < 0$

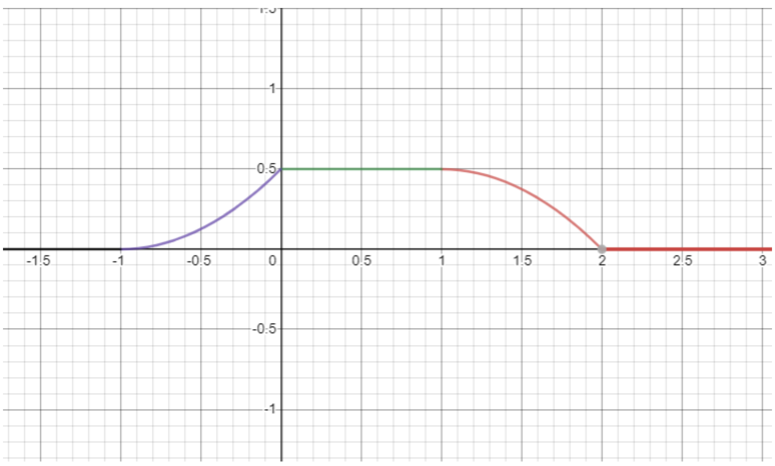
$$\int_0^{t+1} t\ dt$$
$$= \frac{t^2+1}{2} + t$$

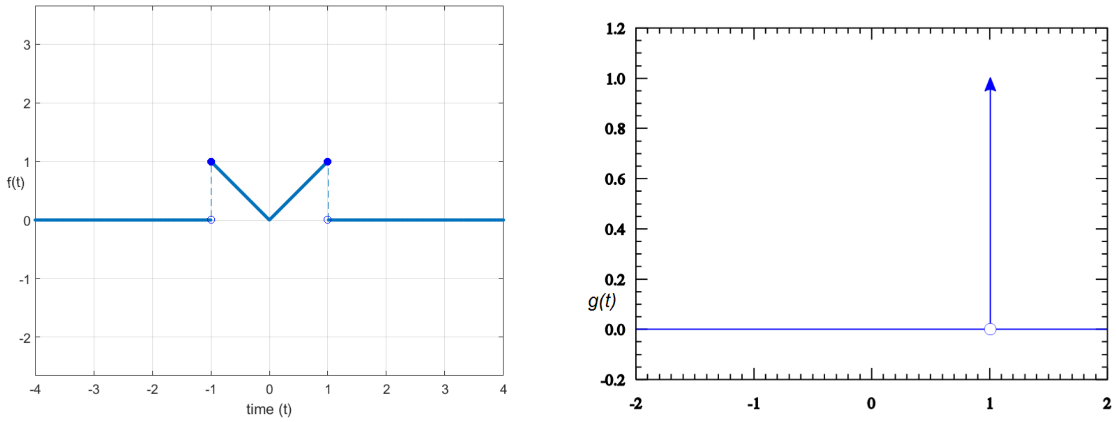
for: $0 < t < 1$

$$\int_0^{1} t\ dt$$
$$= \frac{1}{2}$$

for: $1 < t < 2$

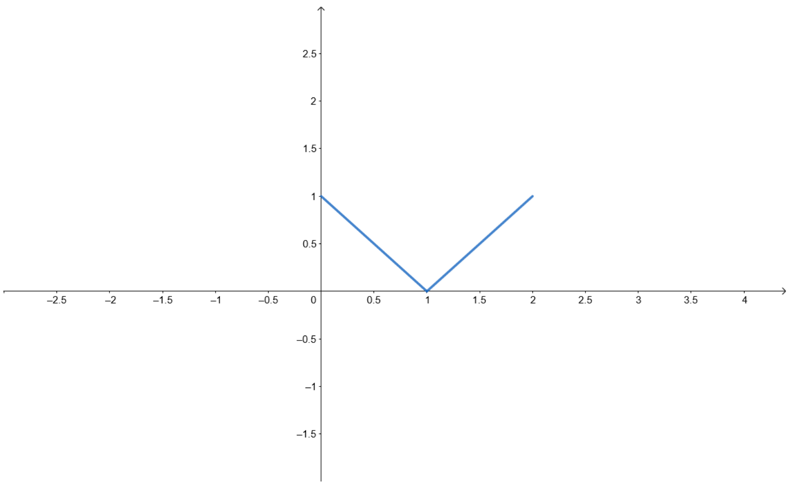
$$\int_{t-1}^{1} -t\ dt$$
$$= -\frac{t^2}{2} + t$$

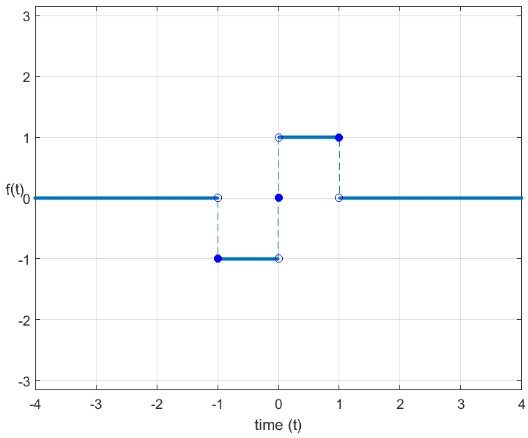


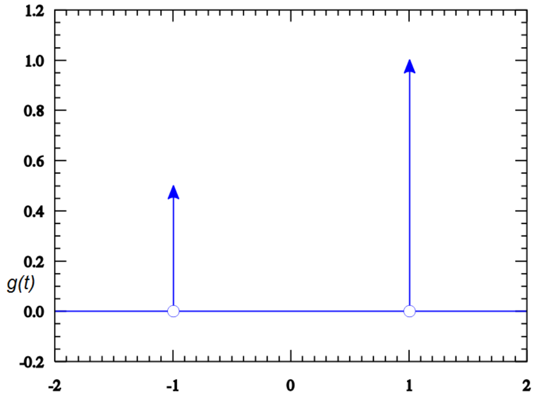


(b)

Solution

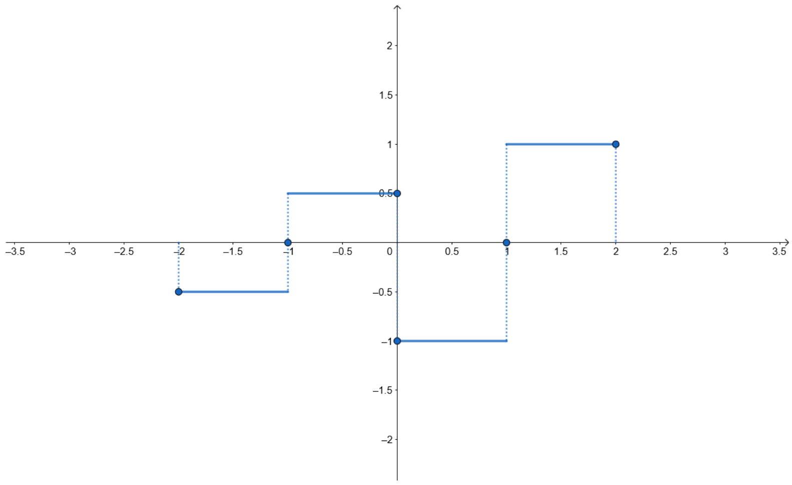


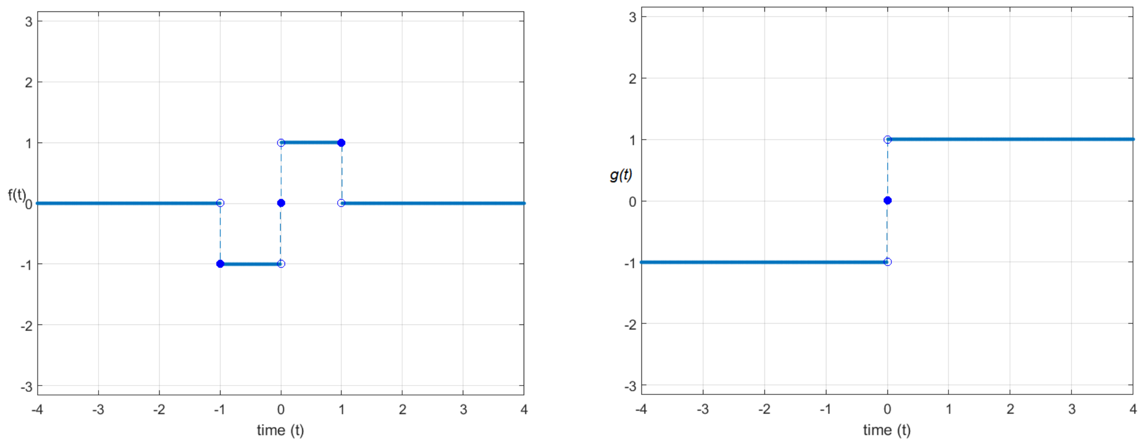




(c)

Solution





(d)

Solution
for: $-\infty < t < -1$

$$\int_{-1}^{0} 1 \ dt + \int_{0}^{1} -1 \ dt$$
$$= 0$$

for: $-1 < t < 0$

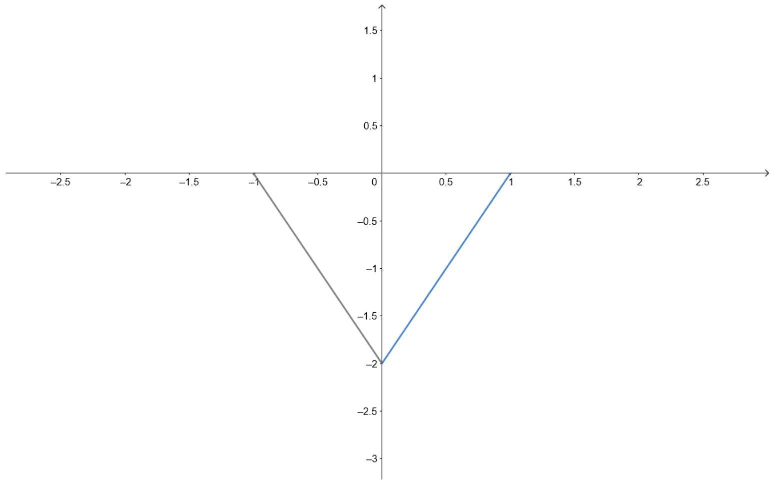
$$\int_{-1}^{t} -1 \ dt + \int_{t}^{0} 1 \ dt + \int_{0}^{1} -1 \ dt$$
$$= -2t - 2$$

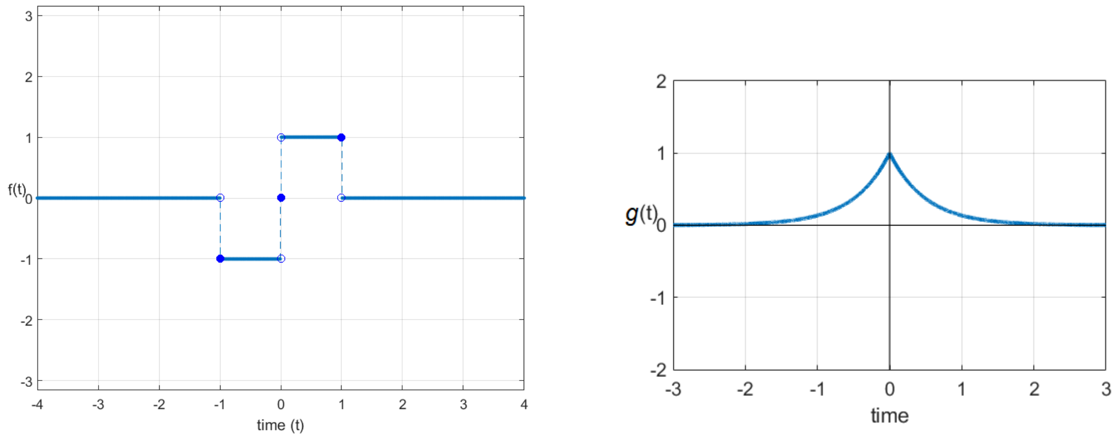
for: $0 < t < 1$

$$\int_{-1}^{0} -1 \ dt + \int_{0}^{t} 1 \ dt + \int_{t}^{1} -1 \ dt$$
$$= 2t - 2$$

for: $1 < t < \infty$

$$\int_{-1}^{0} -1 \ dt + \int_{0}^{1} 1 \ dt$$
$$= 0$$





(e)

Solution

for: $\infty < t < -1$

$$\int_{-1}^{0} (e^{t-\tau})(-1)\ dt + \int_{0}^{1} (e^{t-\tau})(1)\ dt$$
$$= e^{t}(2 - e^{1} - e^{-1})$$

for: $-1 < t < 0$

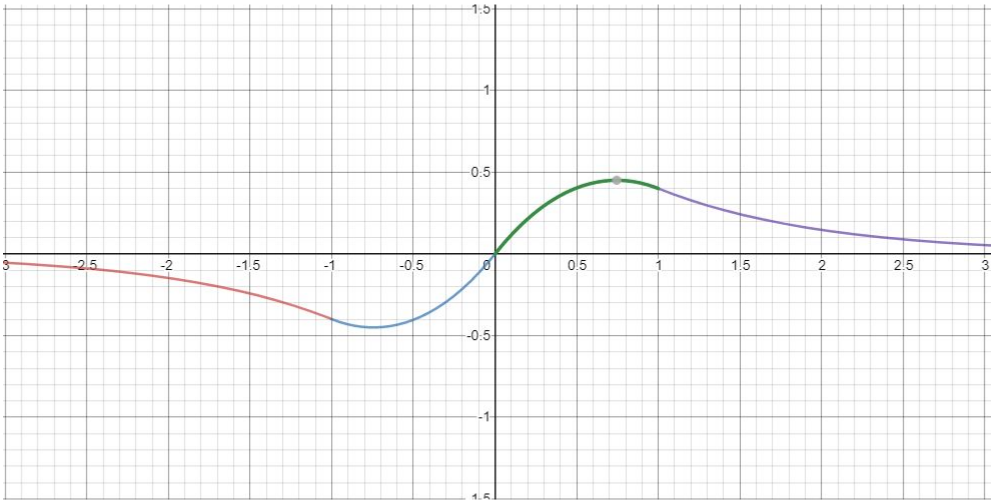
$$\int_{-1}^{t} (e^{-t+\tau})(-1)\ dt + \int_{t}^{0} (e^{t-\tau})(-1)\ dt + \int_{0}^{1} (e^{t-\tau})(1)\ dt$$
$$= e^{t}(-2e^{-t} + 2 - e^{-1}) + e^{-t-1}$$

for: $0 < t < 1$

$$\int_{-1}^{0} (e^{-t+\tau})(-1)\ dt + \int_{0}^{t} (e^{-t+\tau})(1)\ dt + \int_{t}^{1} (e^{t-\tau})(1)\ dt$$
$$= e^{-t}(2e^{t} - 2 + e^{-1}) - e^{t-1}$$

for: $1 < t < \infty$

$$\int_{-1}^{0} (e^{-t+\tau})(-1)\ dt + \int_{0}^{1} (e^{-t+\tau})(1)\ dt$$
$$= e^{-t}(-2 + e^{-1} + e^{1})$$



Problem 2

Use Parseval's theorem (and maybe duality) to evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{\sin^4 2t}{t^4} dt$$

$$\wedge_a = \begin{cases} 1 - \dfrac{|t|}{a}, & |\omega| < a \\ 0, & |\omega| > a \end{cases}$$

From the transforms table, we know that the triangle function with $a = 4$,

$$\begin{cases} 1 - \dfrac{|t|}{a}, & |\omega| < 4 \\ 0, & |\omega| > 4 \end{cases}$$

has Fourier transform equal to $\dfrac{\sin^2 2\omega}{\omega^2}$.

But if the function $f(t) = \dfrac{\sin^2 2t}{t^2}$, using the duality property, its Fourier transform will be

$$F(\omega) = \mathcal{F}\{f(t)\} = 2\pi \begin{cases} 1 - \dfrac{|-\omega|}{4}, & |\omega| < 4 \\ 0, & |\omega| > 4 \end{cases}$$

$$\implies F(\omega) = \begin{cases} 2\pi - \dfrac{\pi}{2}|\omega|, & |\omega| < 4 \\ 0, & |\omega| > 4 \end{cases}$$

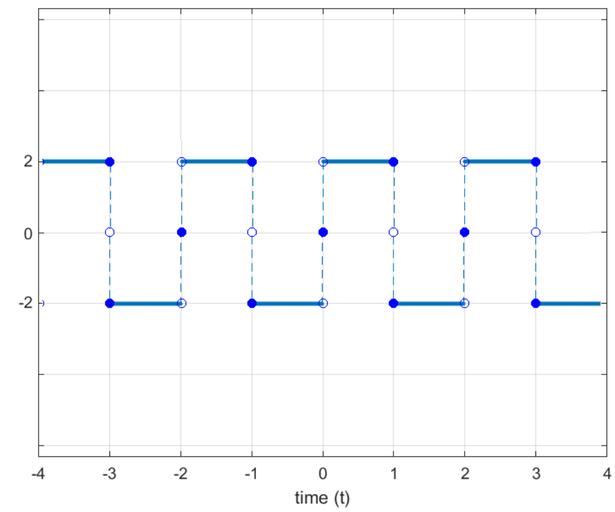
Now using Parseval's theorem,

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

$$\implies \int_{-\infty}^{\infty} \left(\frac{\sin^2 2t}{t^2}\right)^2 dt = \frac{1}{2\pi} \int_{-4}^{4} \left(2\pi - \frac{\pi}{2}|\omega|\right)^2 d\omega = \frac{1}{\pi} \int_{0}^{4} \left(2\pi - \frac{\pi}{2}\omega\right)^2 d\omega = \pi \int_{0}^{4} \left(2 - \frac{\omega}{2}\right)^2 d\omega = \frac{16\pi}{3}$$

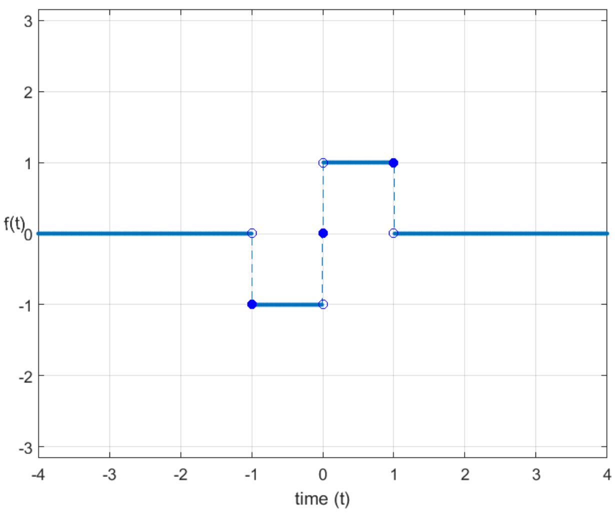
Problem 3

Find the function $g(t)$ whose convolution with $f(t)$ in Problem 1(d) gives the following periodic function.



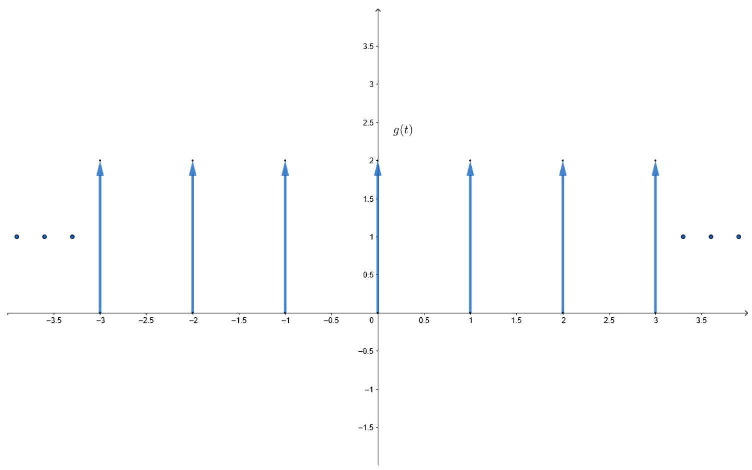
$f(t) * g(t)$

where $f(t)$ is shown in the following figure. Also sketch the graph of $g(t)$.



$f(t)$

Solution



Problem 4

Find the Fourier transform of the following functions and sketch their graphs. You can use the transforms table and properties to directly evaluate the Fourier transforms.

(a) $f(t) = 5e^{-i2t}$

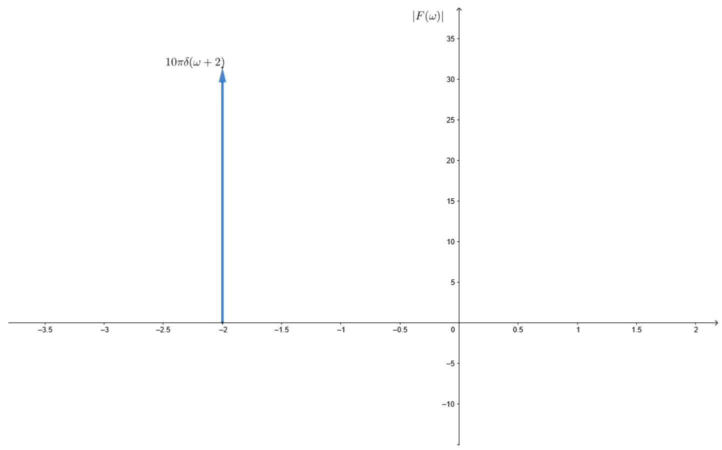
$$\delta t \longrightarrow 1$$

Using duality

$$1 \longrightarrow 2\pi\delta(\omega)$$

$$5 \longrightarrow 10\pi\delta(\omega)\text{frequency-shift}$$

$$5e^{-i2t} \longrightarrow 10\pi\delta(\omega+2)$$



(b) $f(t) = \delta(t-100)$ (Plot the amplitude and phase spectrums)

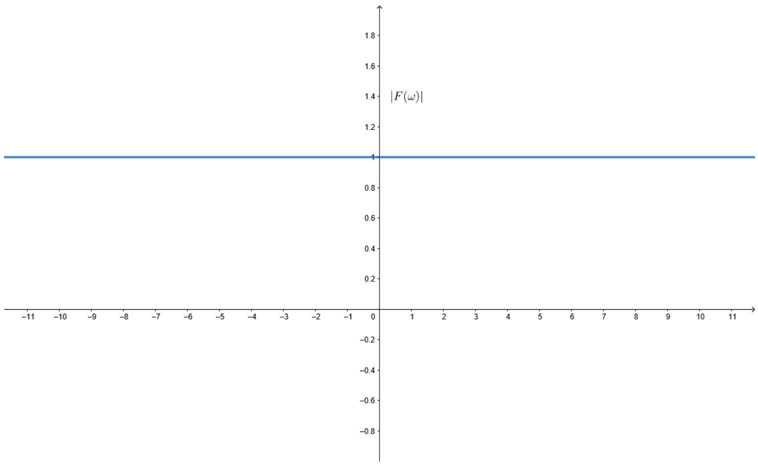
Using the transforms table

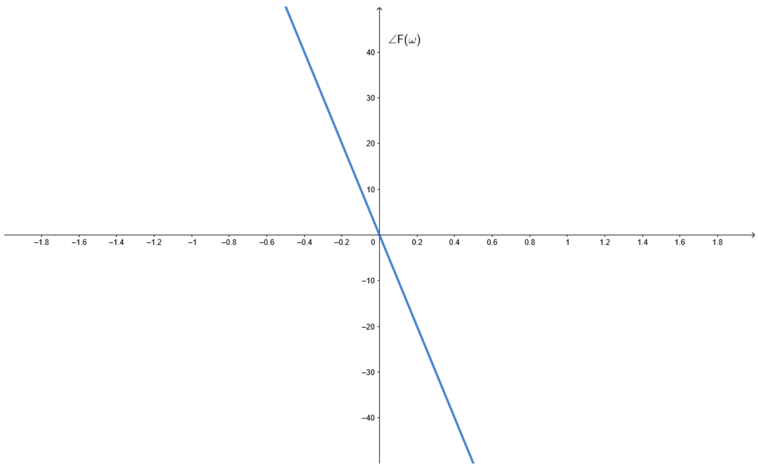
$$\delta(t) \longrightarrow 1$$

time-shift

$$\delta(t-100) \longrightarrow e^{-i100\omega}$$

Comparing $e^{-i100\omega}$ to the complex form $re^{i\theta}$, we can see $r = 1$ and $\theta = -100\omega$





(c) $f(t) = \cos 2t + 3$

Using the transforms table

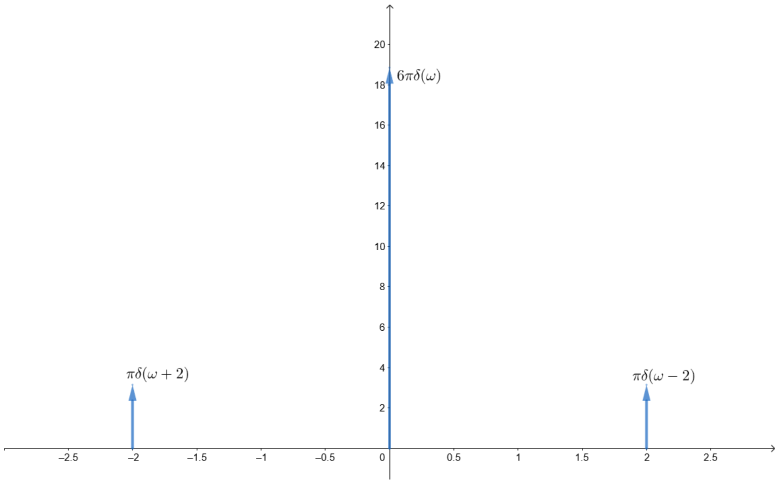
$$\cos 2t \longrightarrow \pi\{\delta(\omega+2) + \delta(\omega-2)\}$$
$$3 \longrightarrow 6\pi\{\delta(\omega)\}$$

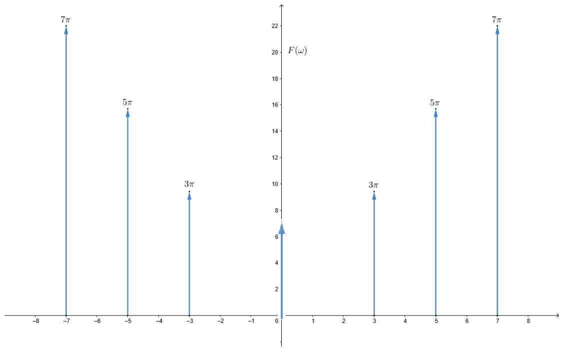
Linearity

$$\cos 2t + 3 \longrightarrow \pi\{\delta(\omega+2) + \delta(\omega-2)\} + 6\pi\{\delta(\omega)\}$$

(d) $f(t) = 1 + 3\cos 3t + 5\cos 5t + 7\cos 7t$

$$\begin{aligned} 1 + 3\cos 3t + 5\cos 5t + 7\cos 7t \longrightarrow\ & 2\pi\delta(\omega) + 3\pi\delta(\omega+3) + 3\pi\delta(\omega-3) \\ & + 5\pi\delta(\omega+5) + 5\pi\delta(\omega-5) \\ & + 7\pi\delta(\omega+7) + 7\pi\delta(\omega-7) \end{aligned}$$





(e) $f(t) = \text{sgn}(5t)$ (Plot the amplitude and phase spectrums)

Using the transforms table

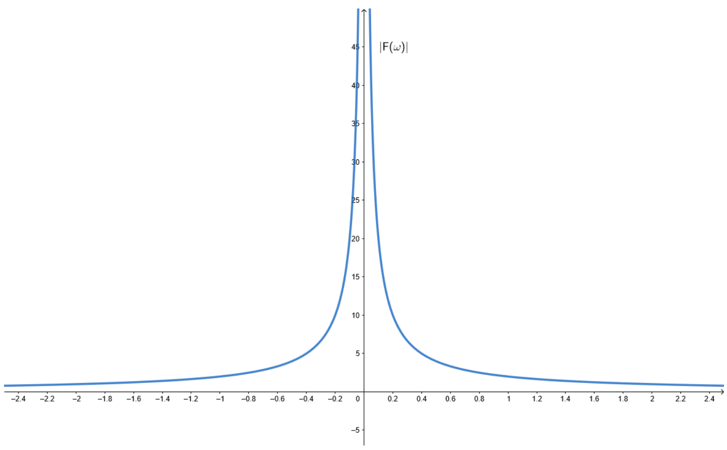
$$\text{sgn}\,(t) \longrightarrow \frac{2}{i\omega}$$

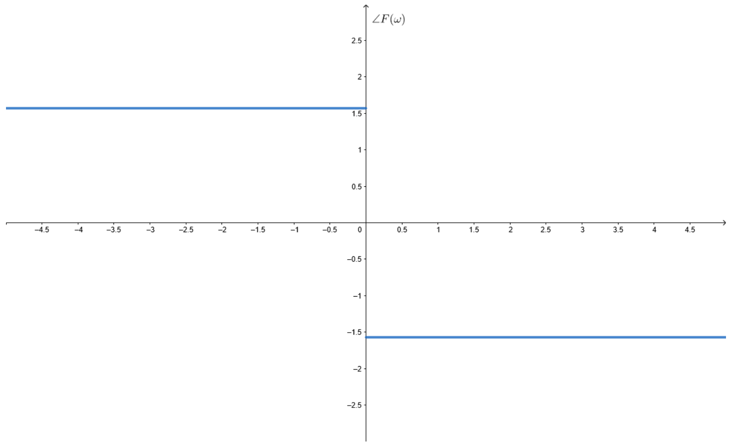
Time scaling

$$\text{sgn}\,(5t) \longrightarrow \frac{1}{5}\frac{2}{i\frac{\omega}{5}}$$

$$\Longrightarrow \frac{2}{i\omega}$$

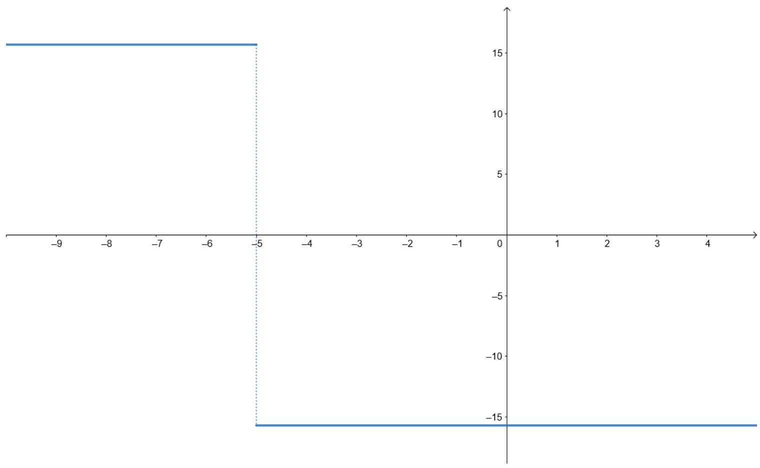
For amplitude plot: $|F(\omega)| = \dfrac{2}{|\omega|}$





(f) $f(t) = \dfrac{5e^{-i2t}}{t}$ (Plot the imaginary part)

$$\begin{aligned} \operatorname{sgn}(t) &\longrightarrow \frac{2}{i\omega} \\ \frac{2}{it} &\longrightarrow -2\pi \operatorname{sgn}(\omega) \\ \frac{1}{t} &\longrightarrow -i\pi \operatorname{sgn}(\omega) \\ \frac{5}{t} &\longrightarrow -i5\pi \operatorname{sgn}(\omega) \\ \frac{5e^{-i2t}}{t} &\longrightarrow -i5\pi \operatorname{sgn}(\omega+5) \end{aligned}$$



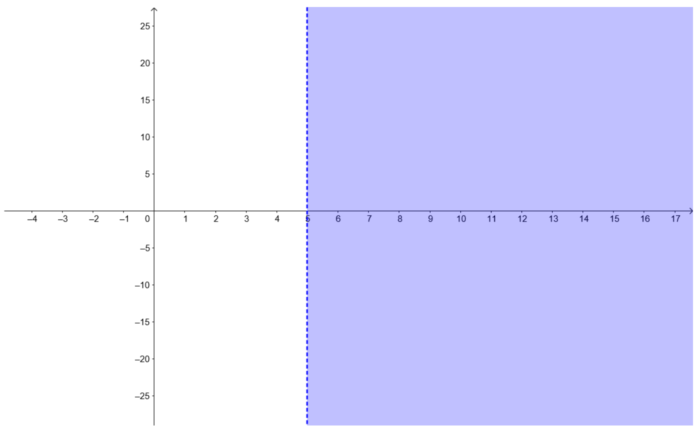
Problem 5

For each of the following functions, if the function is of 'exponential order', find its Laplace transform using transforms table and properties. Also find and sketch the region of absolute convergence in each case. Assume that all the functions are causal i.e. $f(t) = 0$ for $t < 0$.

(a) $3e^{5t}\cos 2t$

$$\cos 2t \longrightarrow \frac{s}{s^2+4}$$
$$3\cos 2t \longrightarrow \frac{3s}{s^2+4}$$
$$3e^{5t}\cos 2t \longrightarrow \frac{3(s-5)}{(s-5)^2+4}$$

For the function to have absolute convergence $\int_0^\infty \left|f(t)e^{-st}\right| < \infty$ the exponent $e^{5t}e^{-st}$ should converge.
From the above condition: $-s+5<0$ and $s>5$
ROC: $s>5$

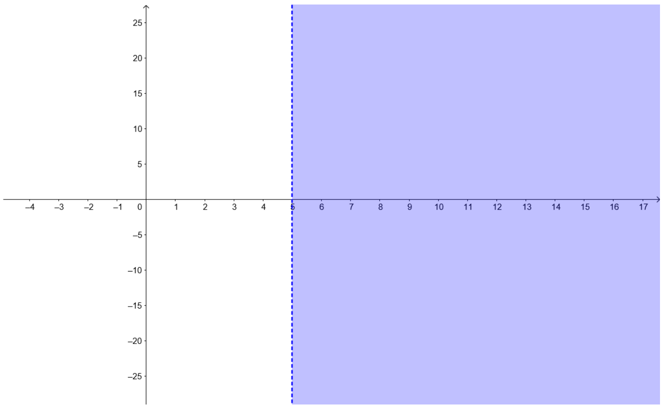


(b) $\int_0^t e^{5\tau} \cos 2\tau d\tau$

Using the integration property:

$$\mathcal{L}\{f(t)\} = F(s)$$
$$\mathcal{L}\left\{\int_0^t f(t)\right\} = \frac{F(s)}{s}$$
$$\mathcal{L}\{3e^{5t} \cos 2t\} = \frac{3(s-5)}{(s-5)^2+4}$$
$$\mathcal{L}\left\{\int_0^t e^{5\tau} \cos 2\tau d\tau\right\} = \frac{3(s-5)}{s(s-5)^2+4s}$$

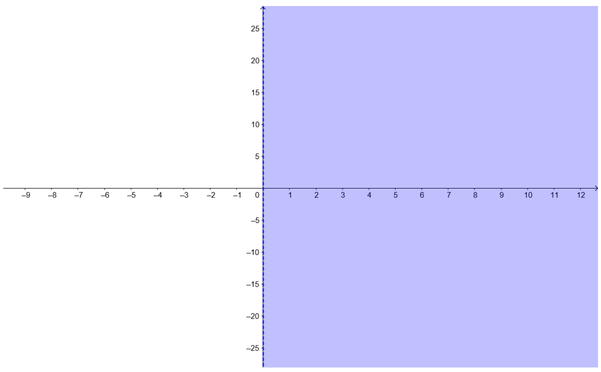
As this is basically the same integral as the first part, so this integral converges for the same value of s.
ROC: $s > 5$



(c) $(t-2)^5$

$$t^n \longrightarrow \frac{n!}{s^{n+1}}$$
$$t^5 \longrightarrow \frac{5!}{s^6}$$
Time-shift
$$(t-2)^5 \longrightarrow \frac{5!e^{-2s}}{s^6}$$

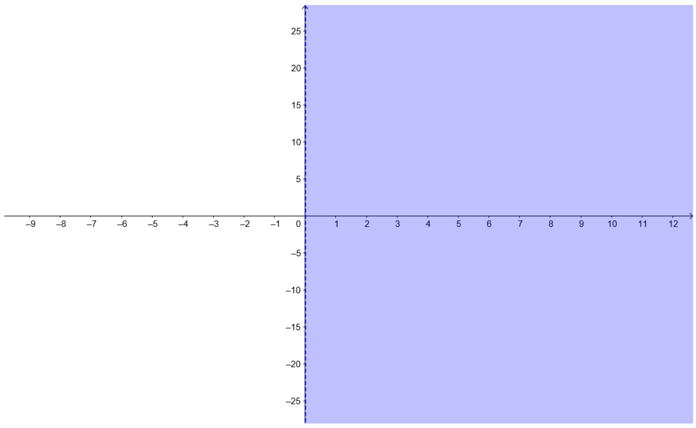
Being an exponential order, we know t^n converges for $s > 0$
ROC: $s > 0$



(d) $u(t-1) - \delta(t-3)$

$$u(t) \longrightarrow \frac{1}{s}$$
$$u(t-1) \longrightarrow \frac{e^{-s}}{s}$$
$$\delta(t) \longrightarrow 1$$
$$\delta(t-3) \longrightarrow e^{-3s}$$
$$u(t-1) - \delta(t-3) \longrightarrow \frac{e^{-s}}{s} - e^{-3s}$$

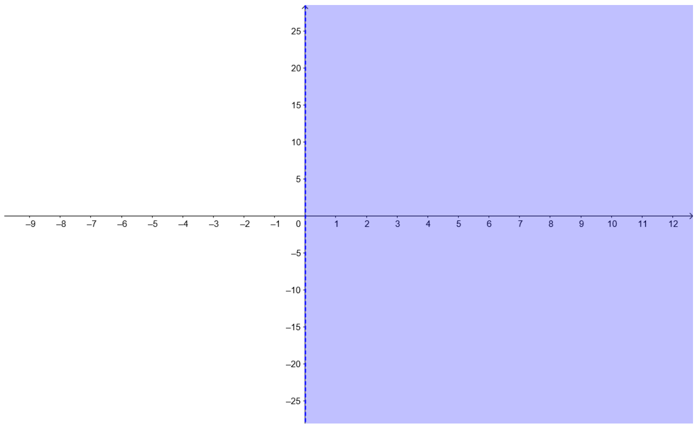
The laplace integral converges for $s > 0$
ROC: $s > 0$



(e) $u(t-1) - u(t-3)$

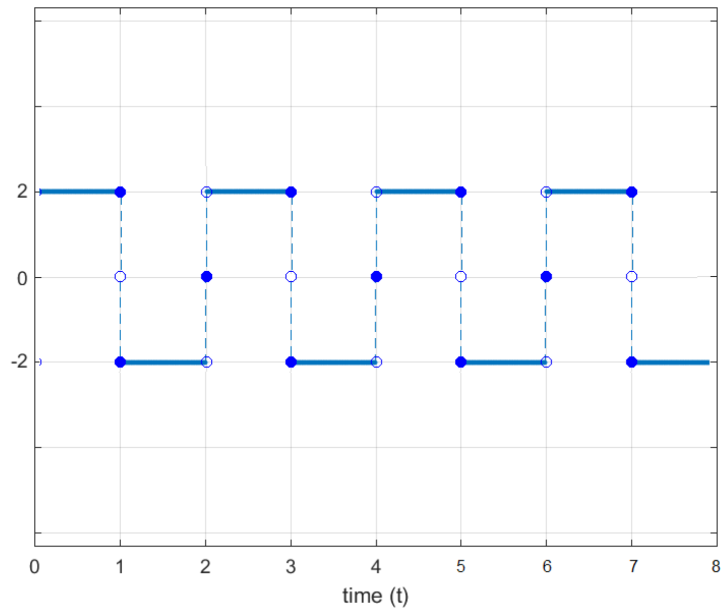
$$u(t-1) \longrightarrow \frac{e^{-s}}{s}$$
$$u(t-3) \longrightarrow \frac{e^{-3s}}{s}$$
$$u(t-1) - u(t-3) \longrightarrow \frac{e^{-s} - e^{-3s}}{s}$$

ROC: $s > 0$



Problem 6

Find Laplace transform of the following periodic function.



Periodic function $f(t)$

Solution

$F(s) = \dfrac{F_T(s)}{1 - e^{-sT}}$

$$F_T(s) = \int_0^t f(t)e^{-st}$$

$$F_T(s) = \int_0^1 2e^{-st} + \int_1^2 -2e^{-st}$$

$$F_T(s) = \frac{-2e^{-st}}{s}|_0^1 + \frac{2e^{-st}}{s}|_1^2$$

$$F_T(s) = \frac{-4e^{-s}}{s} + \frac{2e^{-2s}}{s} + \frac{1}{s}$$

$$F(s) = \frac{\frac{-4e^{-s}+2e^{-2s}+1}{s}}{1 - e^{-2s}}$$

$$F(s) = \frac{-4e^{-s} + 2e^{-2s} + 1}{s - se^{-2s}}$$
