Homework 5 Solution

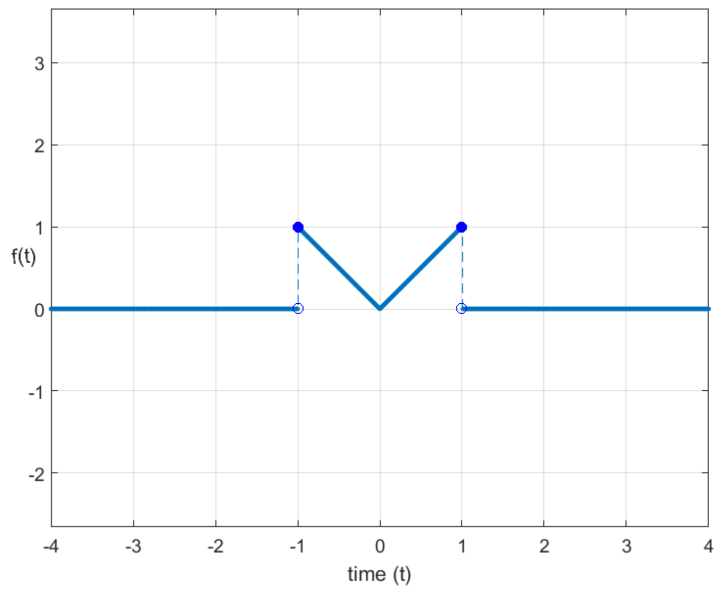
Spring 2019

Problem 1

Find the Fourier transform for each of the following exactly as directed. Also sketch the amplitude spectrum and phase spectrum in each case for $-4\pi < \omega < 4\pi$.

(a) Find the Fourier transform of the following function.

[Hints: Define the function in its appropriate intervals. Use evenness/oddness to simplify the Fourier transform integral.]



$$f(t) = \begin{cases} t, & 0 \leq t \leq 1 \\ -t, & -1 \leq t < 0 \\ 0, & otherwise \end{cases}$$

Even function

$$F(\omega) = 2\int_0^1 t\cos\omega t dt$$

$$F(\omega) = 2\left[\left(\frac{t\sin\omega t}{\omega}\Big|_0^1 + \left(\frac{\cos\omega t}{\omega^2}\Big|_0^1\right]$$

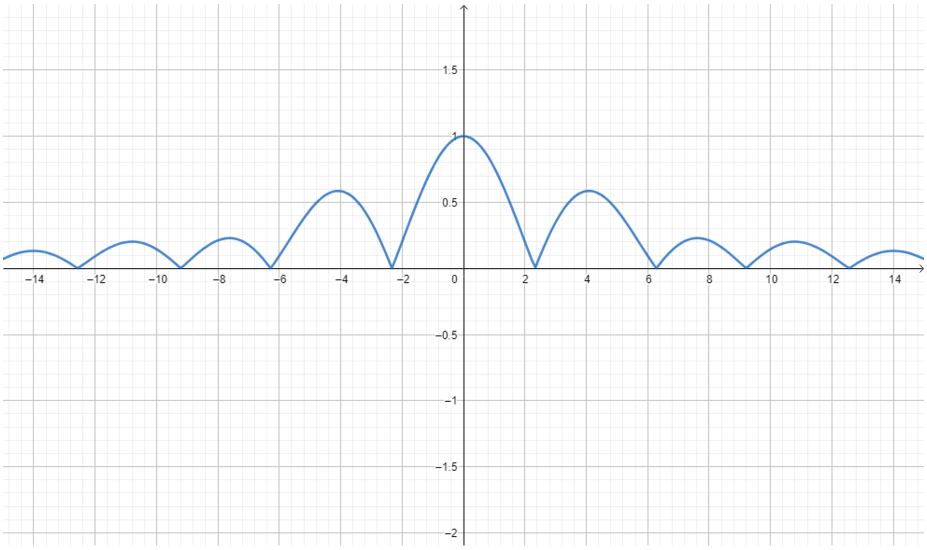
$$F(\omega) = 2\left[\frac{\sin\omega}{\omega} + \frac{\cos\omega}{\omega^2} - \frac{1}{\omega^2}\right]$$

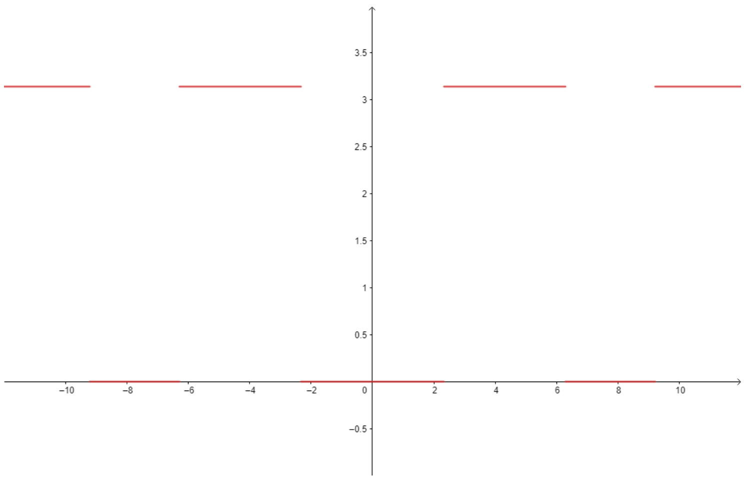
$$F(\omega) = 2\left[\frac{\sin\omega}{\omega} - \frac{2\sin^2\frac{\omega}{2}}{\omega^2}\right]$$

$$F(\omega) = \frac{2\sin\omega}{\omega} - \frac{4\sin^2\frac{\omega}{2}}{\omega^2}$$

$$F(0) = 2\int_0^1 t dt$$

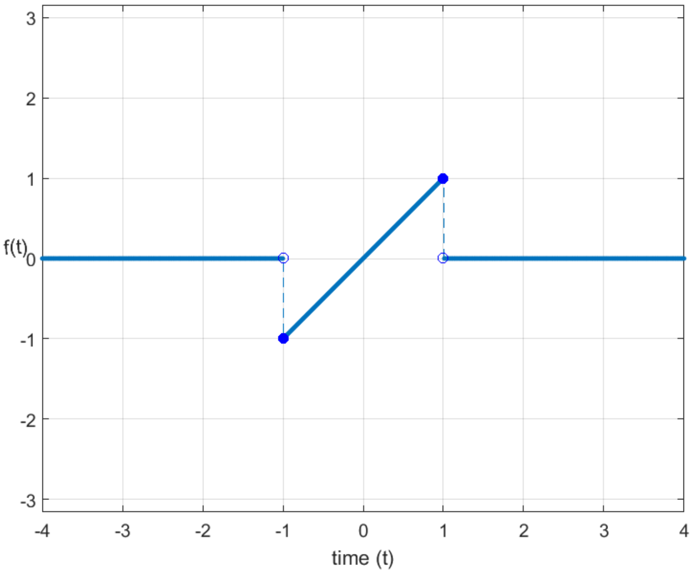
$$F(0) = 1$$





(b) Find the Fourier transform of the following function.

[Hints: Define the function in its appropriate intervals. Use evenness/oddness to simplify the Fourier transform integral.]



$$f(t) = \begin{cases} t, & -1 \leq t \leq 1 \\ 0, & otherwise \end{cases}$$

Odd function

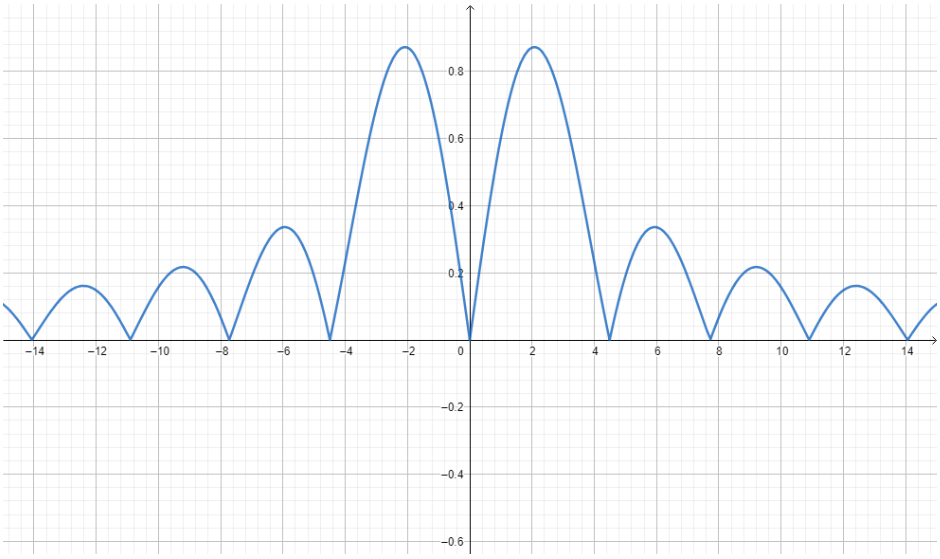
$$F(\omega) = -2i \int_0^1 t \sin \omega t dt$$

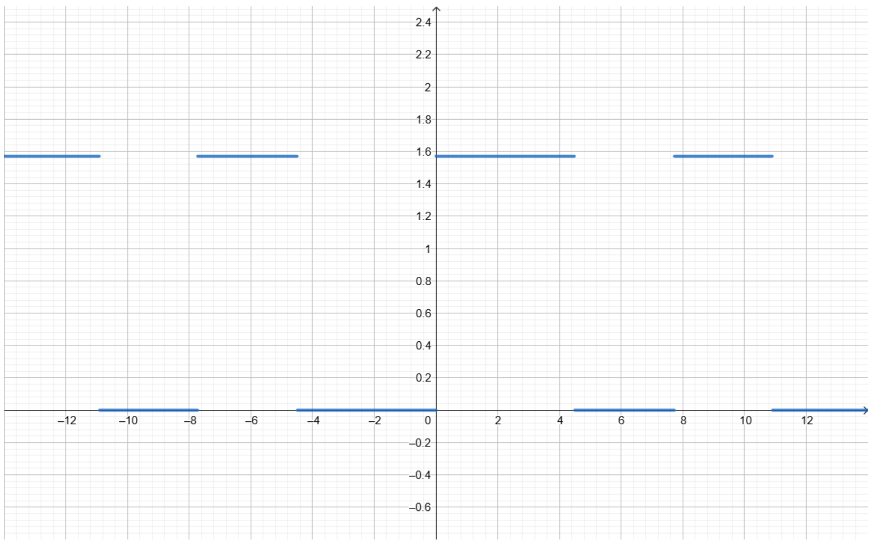
$$F(\omega) = 2\left[\left(\frac{t\cos\omega t}{\omega}\Big|_0^1 + \left(\frac{\sin\omega t}{\omega^2}\Big|_0^1\right]$$

$$F(\omega) = \frac{2i\cos\omega}{\omega} - \frac{2i\sin\omega}{\omega^2}$$

$$F(0) = -2i \int_0^1 t \sin 0 dt$$

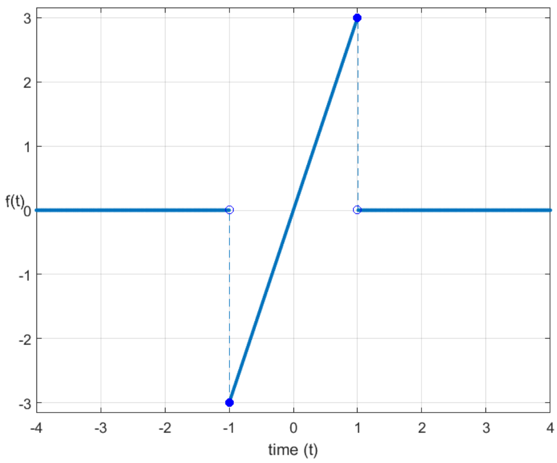
$$F(0) = 0$$





(c) Using your answer to (b), find the Fourier transform of the following function.

[Hint: Linearity]

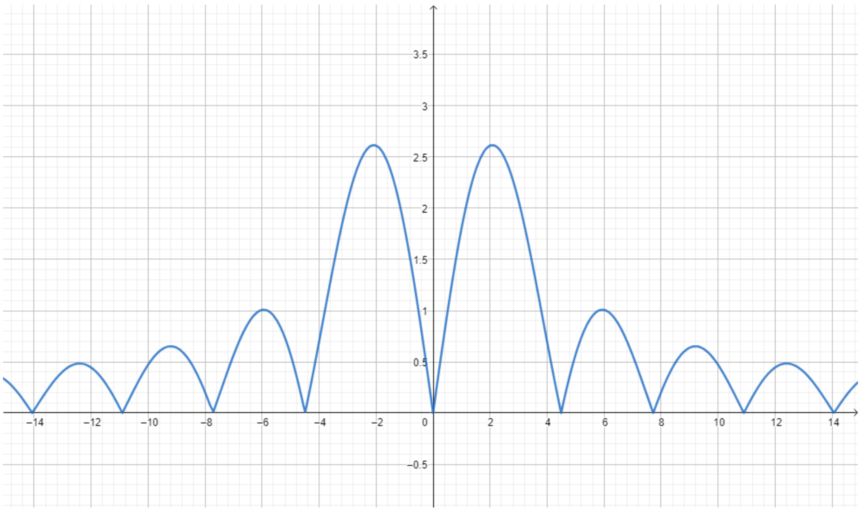


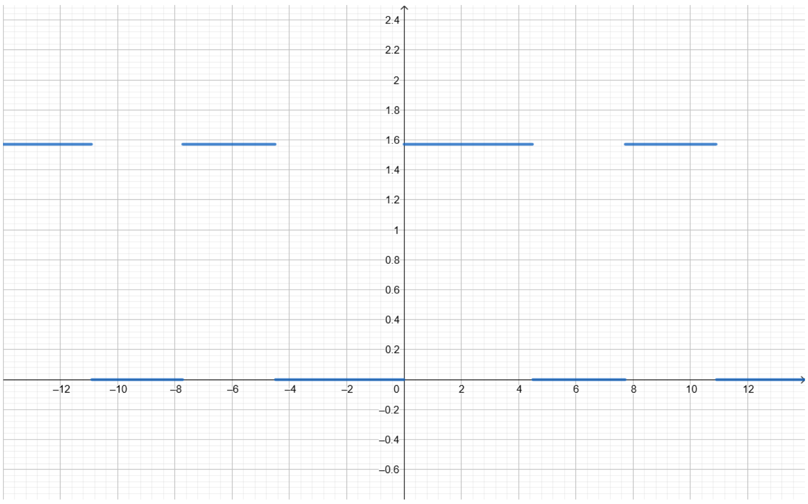
If we call the function in part $(b) f(t)$ and the above provided function $g(t)$. The following relation exists between the two:

$$g(t) = 3f(t)$$

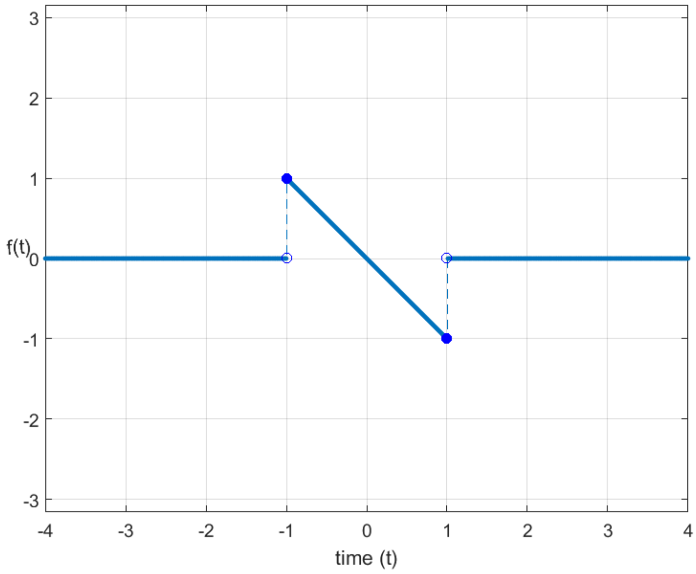
Using the Linearity property

$$G(\omega) = 3F(\omega)$$
$$G(\omega) = 6i\left[\frac{\cos\omega}{\omega} - \frac{\sin\omega}{\omega^2}\right]$$



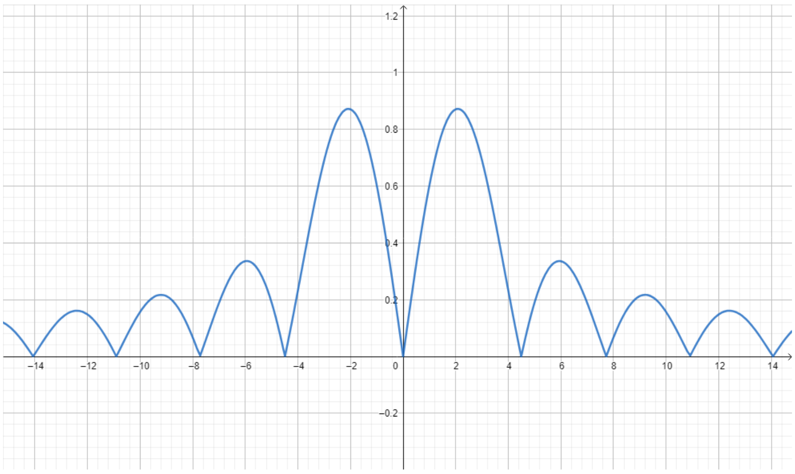


(d) Using your answer to (b), find the Fourier transform of the following function.
[Hint: Linearity or time-reversal]



Using time reversal property:

$$
\begin{aligned}
f(t) &\to F(\omega) \\
f(-t) &\to F(-\omega) \\
F(\omega) &= 2i\left(\frac{\cos\omega}{\omega} - \frac{\sin\omega}{\omega^2}\right) \\
F(-\omega) &= 2i\left(-\frac{\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right)
\end{aligned}
$$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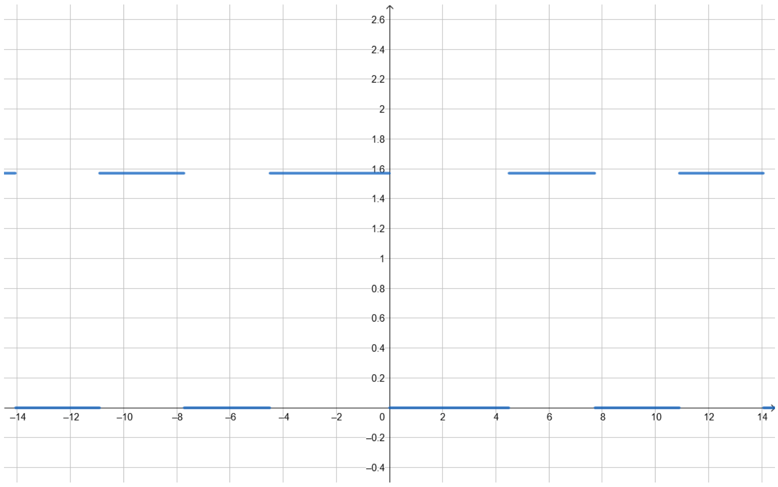
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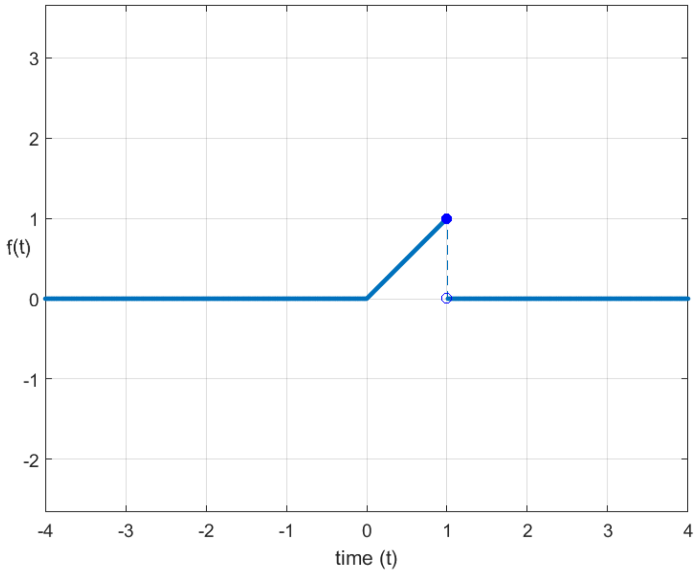
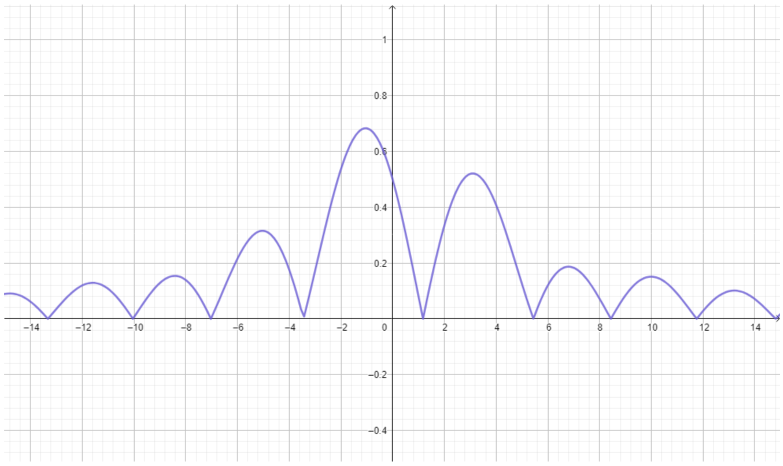
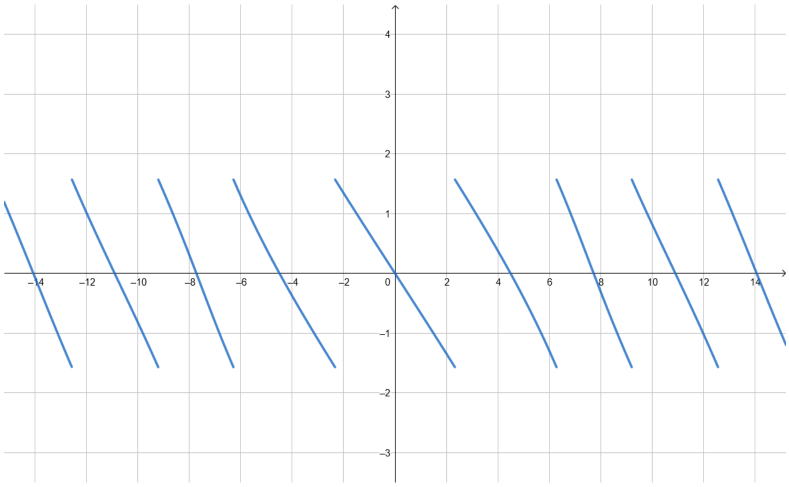
(e) Using your answer to (a) and (b), find the Fourier transform of the following function.
[Hint: Linearity]



Let $f(t)$ be the function in part (a) and $g(t)$ the function in part (b)
$h(t)$, the given function can be obtained the linear operation:

$$h(t) = \frac{1}{2}(f(t) + g(t))$$
$$F(\omega) = \frac{2\sin(\omega)}{\omega} - \frac{4\sin^2(\frac{\omega}{2})}{\omega^2}$$
$$G(\omega) = \frac{2i\cos(\omega)}{\omega} - \frac{2i\sin(\omega)}{\omega^2}$$
$$H(\omega) = \frac{1}{2}(F(\omega) + G(\omega))$$
$$H(\omega) = \frac{\sin(\omega)}{\omega} - \frac{2\sin^2(\frac{\omega}{2})}{\omega^2} + i\frac{\cos(\omega)}{\omega} - i\frac{\sin(\omega)}{\omega^2}$$
$$H(\omega) = \frac{\sin(\omega) + i\cos(\omega)}{\omega} - \frac{2\sin^2(\frac{\omega}{2}) + i\sin(\omega)}{\omega^2}$$

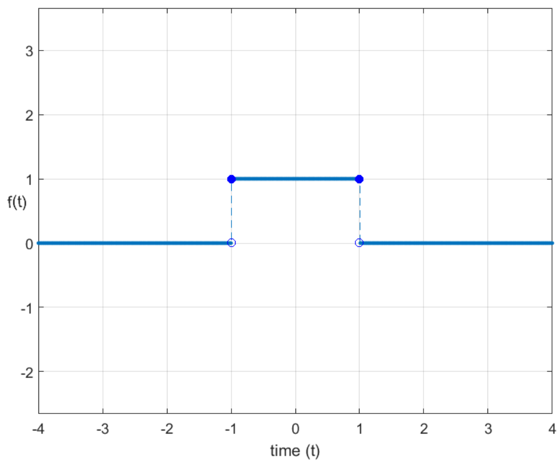




Note(Parts (f) and (g)): Differentiation of a discontinuous function results in some δ function as well appearing at the points of discontinuity, hence extra cos and sin term is added to the function. Performing the Fourier transform using the direct transform formula will yield a result that differs from the one evaluated using the property as the below given function have discontinuities. So for the following parts we'll be ignoring this δ term and use the differentiation property to evaluate the result

(f) Using your answer to (b), find the Fourier transform of the following function.

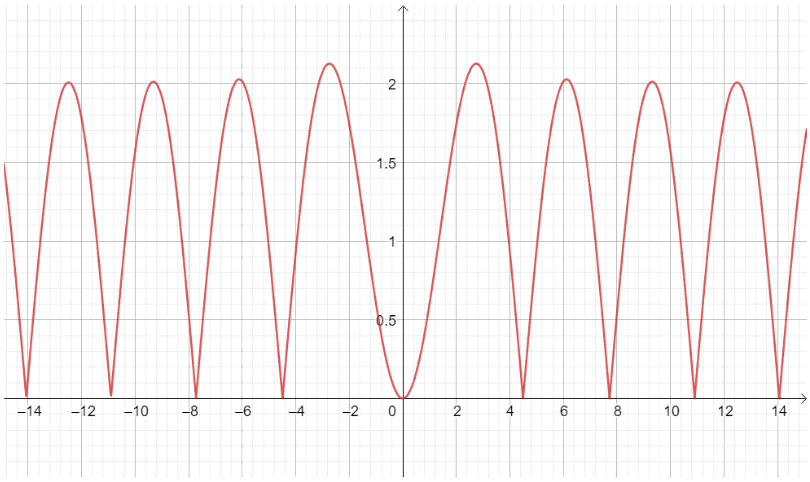
[Hint: Differentiation]

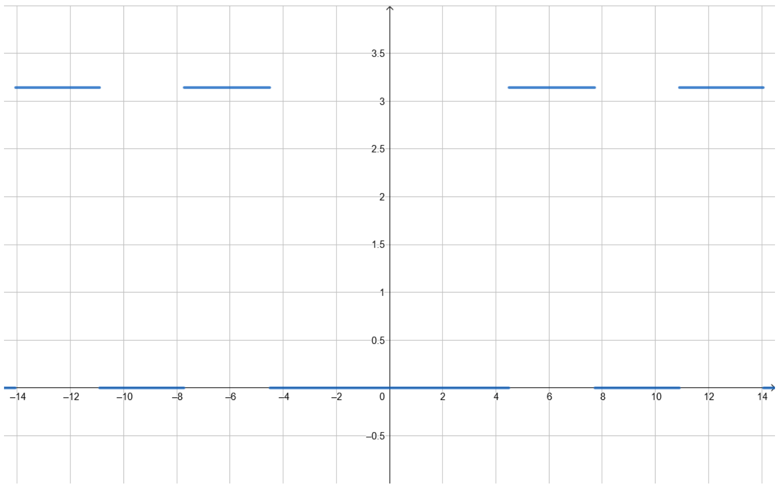


Applying the Differentiation property

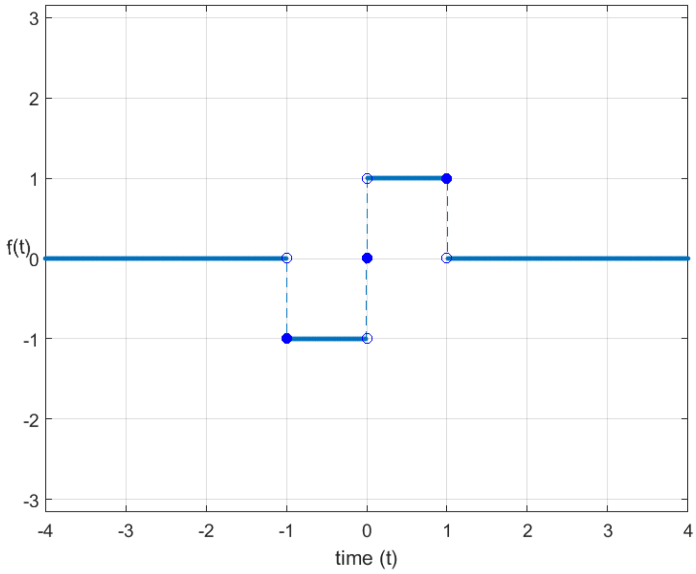
$$f(t) \rightarrow F(\omega)$$
$$f'(t) \rightarrow i\omega F(\omega)$$

$$G(\omega) = i\omega \left[\frac{2i\cos(\omega)}{\omega} - \frac{2i\sin(\omega)}{\omega^2}\right]$$
$$G(\omega) = \frac{2\sin(\omega)}{\omega} - 2\cos(\omega)$$





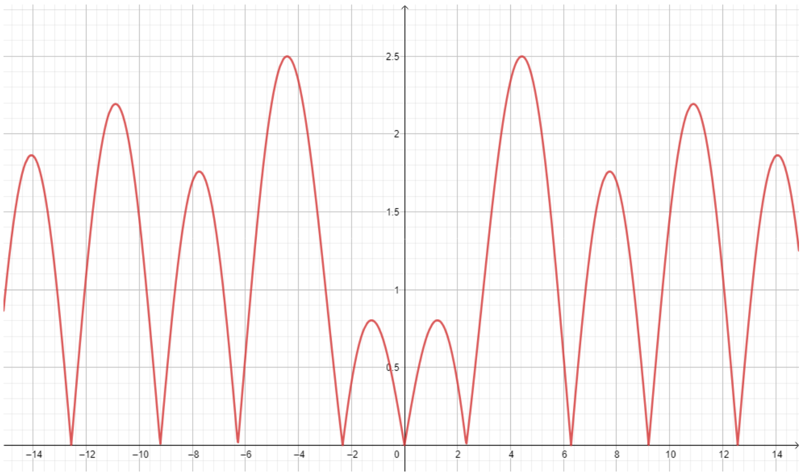
(g) Using your answer to (a), find the Fourier transform of the following function.
[Hint: Differentiation]



Again using the differentiation property and the function $f(t)$ provided in part (a)

$$f(t) \rightarrow F(\omega)$$
$$f(t) \rightarrow \frac{2\sin(\omega)}{\omega} - \frac{4\sin^2\left(\frac{\omega}{2}\right)}{\omega^2}$$

$$f'(t) \rightarrow i\omega F(\omega)$$
$$f'(t) \rightarrow i\omega \frac{2\sin(\omega)}{\omega} - \frac{4\sin^2\left(\frac{\omega}{2}\right)}{\omega^2}$$
$$= i\left(2\sin(\omega) - \frac{4\sin^2\left(\frac{\omega}{2}\right)}{\omega}\right)$$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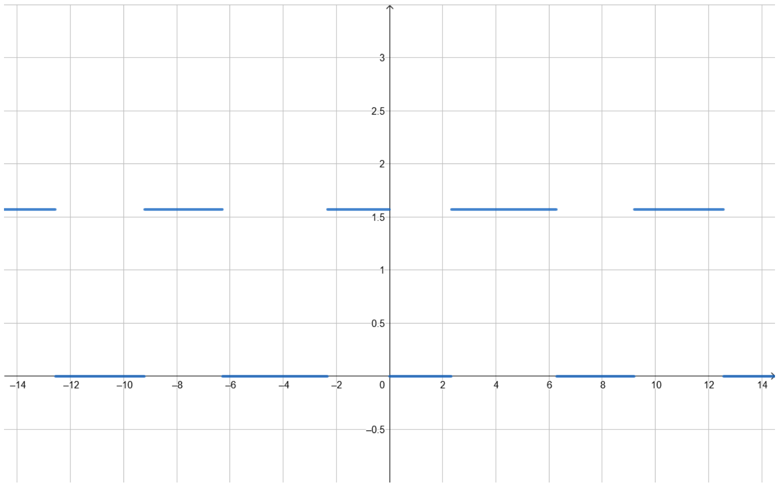
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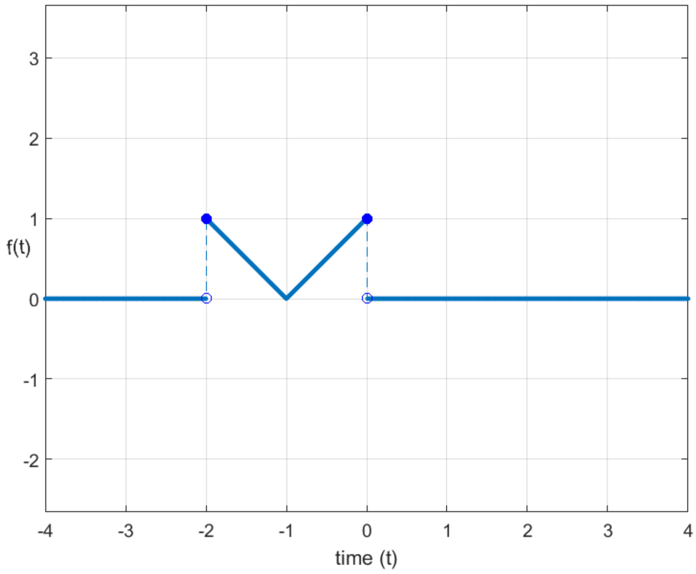
(h) Using your answer to (a), find the Fourier transform of the following function.
[Hint: Time shift]

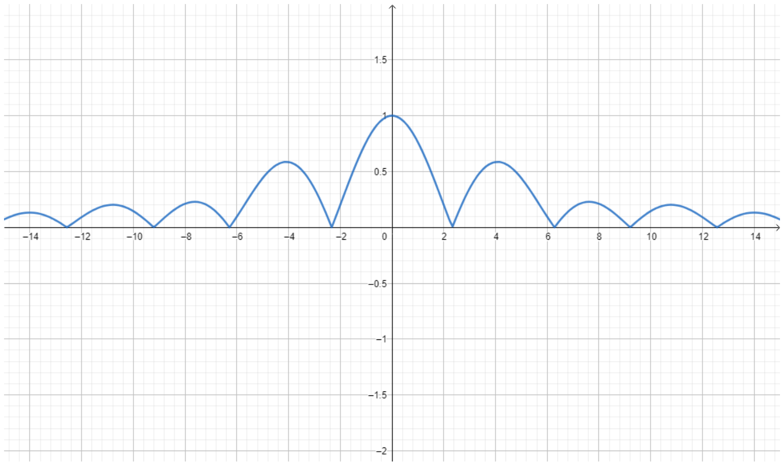


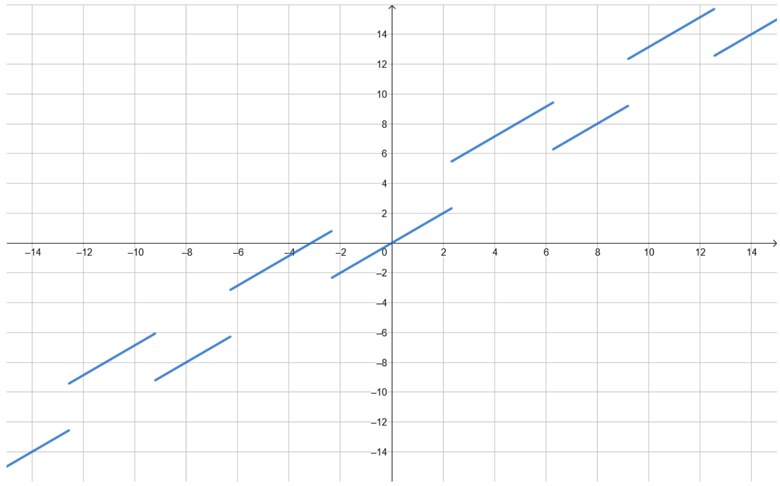
Time shift property

$$f(t) \to F(\omega)$$
$$f(t - t_0) \to e^{-i\omega t_0} F(\omega)$$

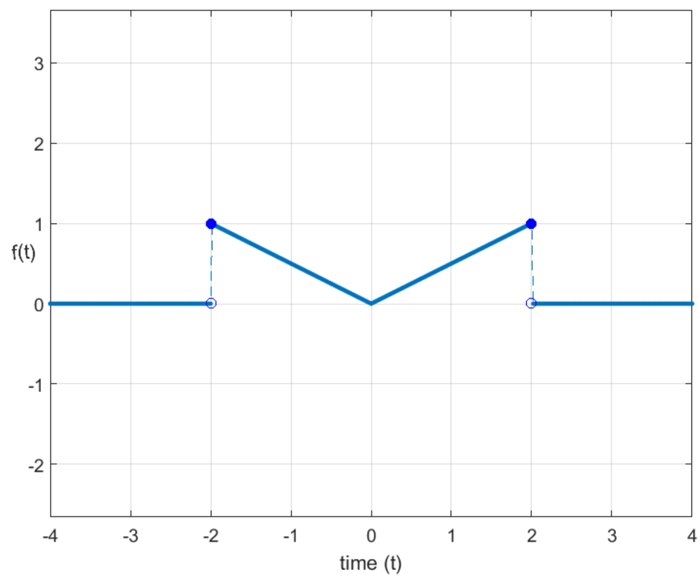
Function in part (a) shifted by 1, i.e $f(t+1)$. So for $t_0 = -1$

$$G(\omega) = e^{i\omega} F(\omega)$$
$$G(\omega) = e^{i\omega} \left(\frac{2\sin(\omega)}{\omega} - \frac{4\sin^2(\frac{\omega}{2})}{\omega^2} \right)$$





(i) Using your answer to (a), find the Fourier transform of the following function.
[Hint: Time scaling]

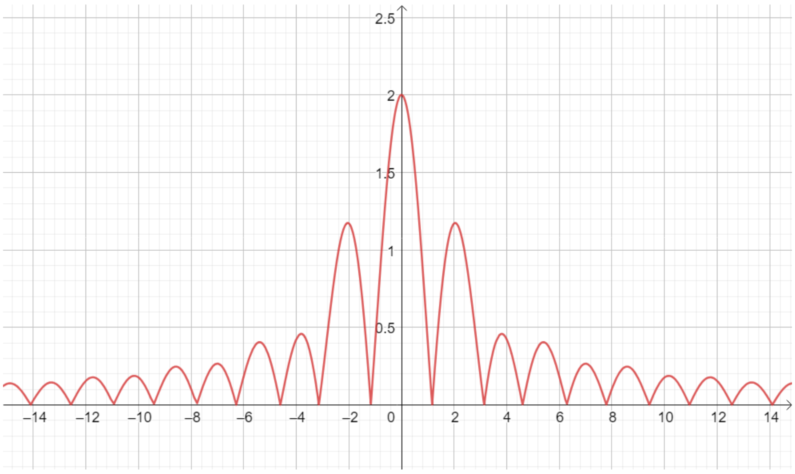


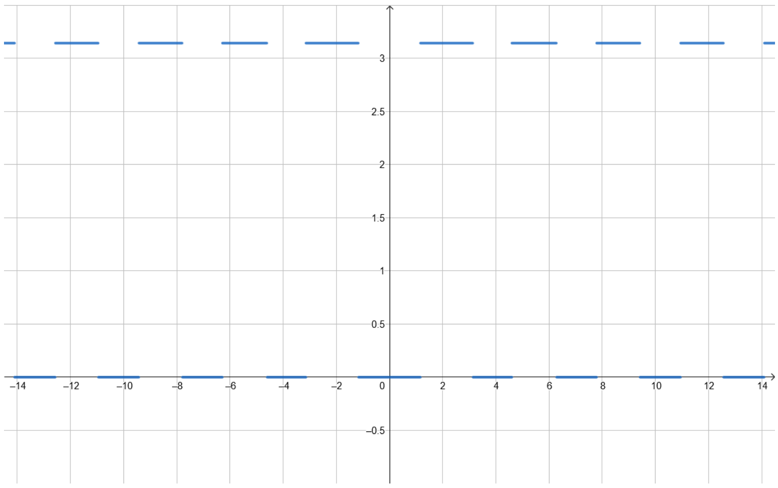
Time scaling property

$$f(t) \to F(\omega)$$
$$f(ct) \to \frac{1}{|c|}F(\frac{\omega}{c})$$

The function provided is a scaled version of the function $f(t)$ from part (a). Where $g(t) = f(\frac{t}{2}$

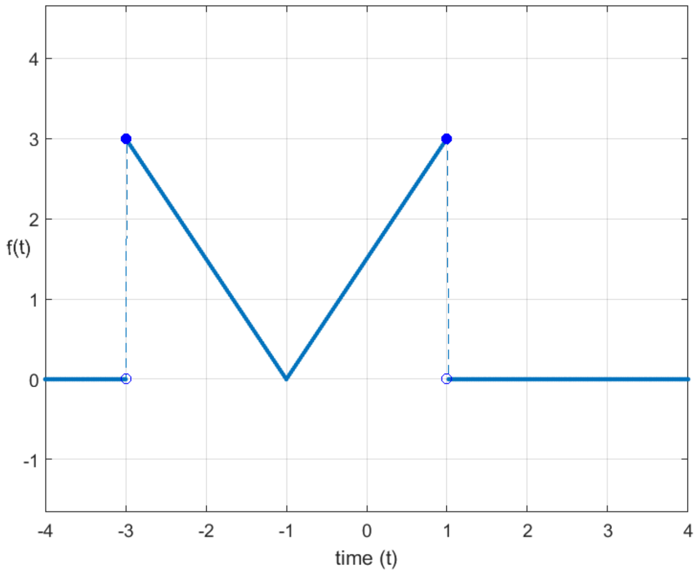
$$f(\frac{t}{2}) \to 2F(2\omega)$$
$$G(\omega) = 2\left[\frac{2\sin(2\omega)}{2\omega} - \frac{4\sin^2(\frac{2\omega}{2})}{4\omega^2}\right]$$
$$G(\omega) = \frac{2\sin(2\omega)}{\omega} - \frac{2\sin^2(\omega)}{\omega^2}$$





(j) Using your answer to (a), find the Fourier transform of the following function.

[Hint: Various properties]



This function $g(t)$ can be obtained from the function $f(t)$ of part (a) by applying the following properties:
Time shift + Time scaling + Linearity

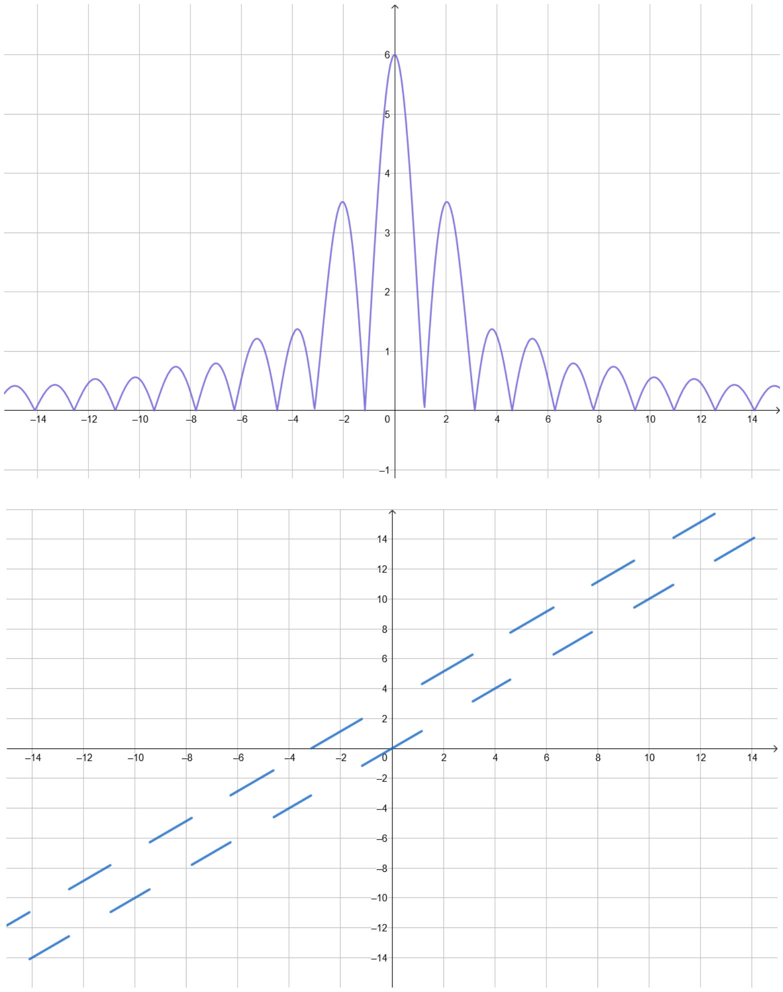
$$g(t) = 3f\left(\frac{t}{2}+1\right)$$

$$f(t) \rightarrow \frac{2\sin(\omega)}{\omega} - \frac{4\sin^2\left(\frac{\omega}{2}\right)}{\omega^2}$$

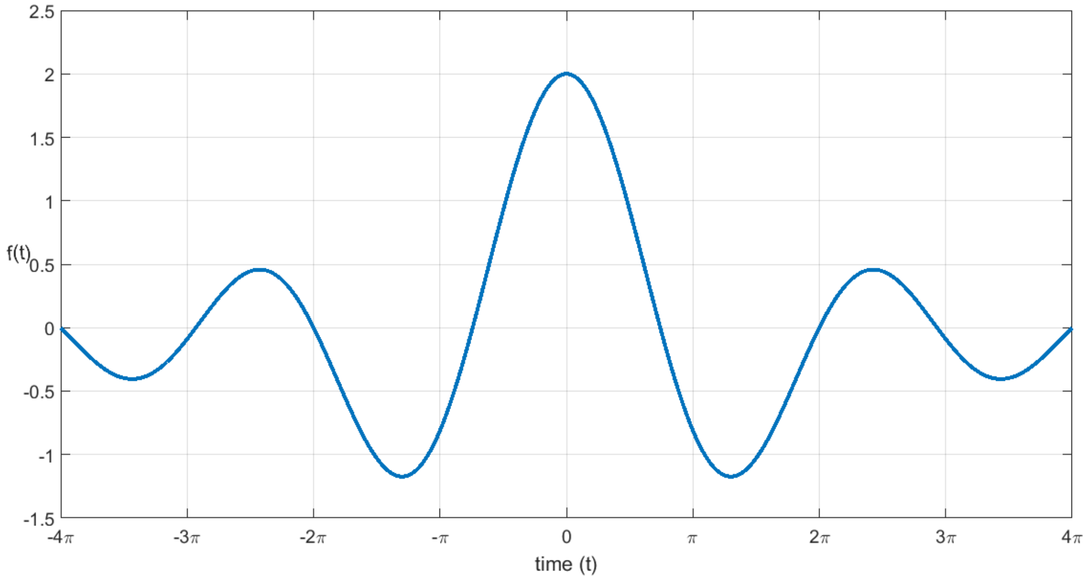
$$f\left(\frac{t}{2}\right) \rightarrow \frac{2\sin(2\omega)}{\omega} - \frac{2\sin^2(\omega)}{\omega^2}$$

$$f\left(\frac{t}{2}+1\right) \rightarrow e^{i\omega}\left(\frac{2\sin(2\omega)}{\omega} - \frac{2\sin^2(\omega)}{\omega^2}\right)$$

$$3f\left(\frac{t}{2}+1\right) \rightarrow 6e^{i\omega}\left(\frac{\sin(2\omega)}{\omega} - \frac{\sin^2(\omega)}{\omega^2}\right)$$



(k) Using your answer to (a), find the Fourier transform of the following function.
[Hint: Duality and Linearity]



Duality property

$$f(t) \to F(\omega)$$
$$F(t) \to 2\pi f(-\omega)$$

The function given above is also multiplied by 2, i.e $2F(t)$.
Hence using linearity and duality property we get:

$$2F(t) \to 2(2\pi f(-\omega))$$
$$2F(t) \to 4\pi f(-\omega)$$
$$F(\omega) = \begin{cases} -4\pi\omega, & -1 \leq \omega < 0 \\ 4\pi\omega, & 0 < \omega \leq 1 \\ 0, & otherwise \end{cases}$$