MT240: Complex Variables and Transforms

Homework 4 Solution

Spring 2018

Problem 1

For each of the following functions

(a) $f(z) = \dfrac{z}{(z-1)(z-2)(z+2)^2}$

(i) Find all the singular points.
Singular points are $z = 1$ and $z = \pm 2$

(ii) Classify each singular point as removable, pole (multiplicity n) or essential. Note: A simple pole has multiplicity $n = 1$.
At $z = 1$

$$\lim_{z\to 1} \frac{z}{(z-2)(z+2)^2} = -\frac{1}{9}$$

At $z = 2$

$$\lim_{z\to 2} \frac{z}{(z-1)(z+1)^2} = \frac{1}{8}$$

Simple Pole at $z = 1$ and $z = 2$
At $z = -2$

$$\lim_{z\to -2} \frac{z}{(z-1)(z-2)} = -\frac{1}{6}$$

Pole of multiplicity 2 at $z = -2$

(iii) Find the residue corresponding to each singular point.

At $z = 1$

$$Res(f,1) = \lim_{z\to 1} \frac{z}{(z-2)(z+2)^2} = -\frac{1}{9}$$

At $z = 2$

$$Res(f,2) = \lim_{z\to 2} \frac{z}{(z-1)(z+1)^2} = \frac{1}{8}$$

At $z = -2$

$$Res(f,-2) = \frac{d}{dz}\left((z-(-2))^2 f(z)\right)$$
$$\lim_{z\to -2} \frac{d}{dz}\left(\frac{z}{(z-1)(z-2)}\right) = -\frac{1}{72}$$

(iv) Evaluate the integral $\oint_{|z|=5} f(z)dz$.

$$\oint_{|z|=5} f(z)dz$$

$$\oint_{|z|=5} f(z)dz = 2\pi i(\text{Res}(f,1) + \text{Res}(f,2) + \text{Res}(f,-2)$$

$$= 2\pi i\left(-\frac{1}{9} + \frac{1}{8} - \frac{1}{72}\right)$$

$$\oint_{|z|=5} f(z)dz = 0$$

(b) $f(z) = \dfrac{\cos 2z - 1}{2z(z+i)}$

(i) Singular points are $z = 0$ and $z = -i$

(ii) Classify each singular point as removable, pole (multiplicity n) or essential. Note: A simple pole has multiplicity $n = 1$.
At $z = 0$

$$\lim_{z \to 0} \frac{\cos(2z) - 1}{(2)(z+i)} = 0$$

Removable at $z = 0$ At $z = -i$

$$\lim_{z \to -i} \frac{\cos(2z) - 1}{2z} = -\frac{\cos(2i) - 1}{2i}$$

Simple Pole at $z = -i$

(iii) Find the residue corresponding to each singular point.

At $z = 0$

$$\text{Res}(f,0) = 0$$

At $z = -i$

$$\text{Res}(f,0) = \lim_{z \to -i} \frac{cos(2z) - 1}{(2z)} = -\frac{cos(2i) - 1}{2i}$$

(iv) Find the residue corresponding to each singular point.

$$\oint_{|z|=5} f(z)dz$$

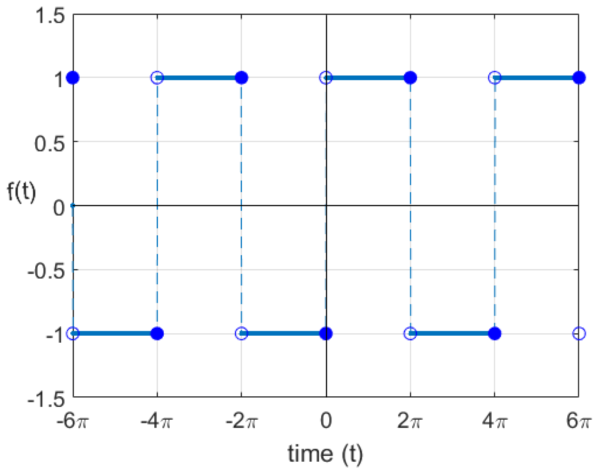
$$\oint_{|z|=5} f(z)dz = 2\pi i(\text{Res}(f,0) + \text{Res}(f,-i))$$

$$= 2\pi i(-\frac{cos(2i) - 1}{2i})$$

$$\oint_{|z|=5} f(z)dz = -\pi(cos(2i) - 1)$$

Problem 2

(a) Consider the periodic function shown in the following figure.



(i) Find its time period T and angular frequency ω_0.
$T = 4\pi$ and $\omega_0 = \frac{1}{2}$

(ii) Is it an even function or an odd function?
Odd function, as $f(-t) = -f(t)$

(iii) Based on (ii), determine whether its complex Fourier series coefficients c_n be real or pure imaginary? Explain.
$$c_n = \frac{a_n - ib_n}{2}$$
c_n will be pure imaginary because the function is odd and it's Fourier series don't have real terms i.e a_n

(iv) Based on (ii), determine the relationship between c_n and c_{-n}.
$c_{-n} = -c_n$

(v) Write down the equation of the function in the interval $-\frac{T}{2} < t \leq \frac{T}{2}$.
$$f(t) = \begin{cases} 1, & 0 < t \leq 2\pi \\ -1, & -2\pi < t \leq 0 \\ 0, & otherwise \end{cases}$$

(vi) Now find its complex Fourier series. Clearly write the final expressions for c_0 and c_n.

$$c_0 = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt$$

$$c_0 = \frac{1}{4\pi}\left(\int_{-2\pi}^{0} -1dt + \int_{0}^{2\pi} 1dt\right) \quad = \quad 0$$

$$c_n = \frac{1}{4\pi}\left(\int_{-2\pi}^{0} -e^{-i\frac{nt}{2}}dt + \int_{0}^{2\pi} e^{-i\frac{nt}{2}}dt\right)$$

$$= \frac{1}{4\pi}\left(\frac{2e^{-i\frac{nt}{2}}}{in}\Bigg|_{-2\pi}^{0} + \frac{2e^{-i\frac{nt}{2}}}{in}\Bigg|_{0}^{2\pi}\right)$$

$$= \frac{1}{4pi}\left(\frac{2-2\cos n\pi}{in} + \frac{-2\cos n\pi + 2}{in}\right)$$

$$c_n = \frac{1-(-1)^n}{in\pi}$$

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1-(-1)^n}{in\pi}e^{i\frac{nt}{2}}$$

(vii) Using your answer to (vi), evaluate the Fourier series coefficients a_n and b_n.

$$c_n = \frac{a_n - ib_n}{2}$$

$$a_n = 0$$

$$b_n = 2ic_n$$

$$b_n = 2i\left(\frac{1-(-1)^n}{in\pi}\right)$$

$$b_n = \frac{2(1-(-1)^n)}{n\pi}$$

(viii) Plot its amplitude spectrum and phase spectrum for $-7 \leq n \leq 7$.

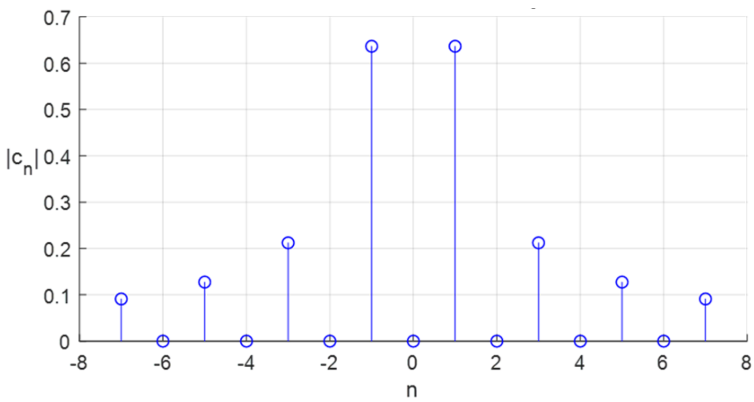


Figure 1: Amplitude Spectrum

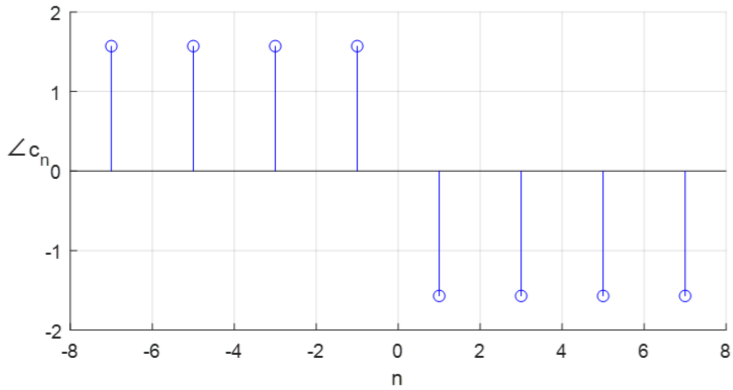


Figure 2: Phase Spectrum

(ix) Calculate the average power in $f(t)$.

$$P_{avg} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt$$

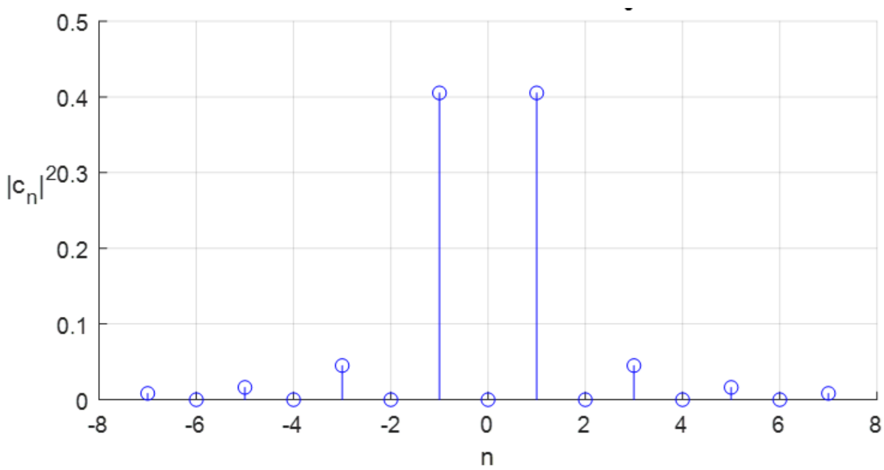
$$P_{avg} = \left(\int_{-2\pi}^{2\pi} (1)^2 dt\right)$$

$$P_{avg} = \frac{1}{4\pi}\left[t\right|_{-2\pi}^{2\pi}$$

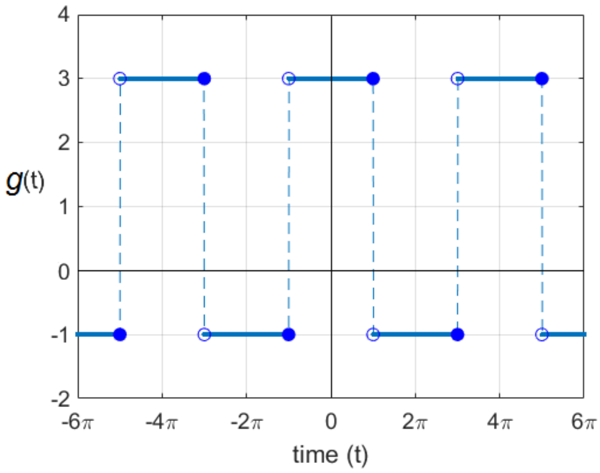
$$P_{avg} = \frac{1}{4\pi}(4\pi)$$

$$P_{avg} = 1$$

(x) Plot its power spectrum $|c_n|^2$ for $-7 \leq n \leq 7$.



(b) Now consider the periodic function $g(t)$ shown in the following figure. This function is somehow related to $f(t)$ from part (a).



(i) Find its time period T and angular frequency ω_0.
$T = 4\pi$ and $\omega_0 = \frac{1}{2}$

(ii) Is it an even function or an odd function?
Even function, as $f(-t) = f(t)$

(iii) Based on (ii), determine whether its complex Fourier series coefficients d_n be real or pure imaginary? Explain.
$d_n = \dfrac{a_n - ib_n}{2}$
d_n will be real. Because the function is even hence it's Fourier series will also be even and real also as $b_n = 0$

(iv) Based on (ii), determine the relationship between d_n and d_{-n}.
$d_{-n} = d_n$

(v) Determine the relationship between $f(t)$ and $g(t)$ and write down an expression for $g(t)$ in terms of $f(t)$. [Hint: $g(t) = af(b(t - t_0)) + c$. Determine a, b, c and t_0.]
$g(t) = 2f(t + \pi) + 1$

(vi) Now using the Fourier series coefficient of $f(t)$, find the complex Fourier series of $g(t)$. Clearly write the final expressions for d_0 and d_n. [Hint: Use properties of Fourier series.]

$$c_n = \frac{1-(-1)^n}{in\pi}$$

Using time shift property:

$$f(t) \rightarrow c_n$$
$$f(t+\pi) \rightarrow c_n e^{-in\omega_0 \pi}$$

And linearity property:

$$f(t) \rightarrow c_n$$
$$2f(t) \rightarrow 2c_n$$

$$g(t) = d_0 + \sum_{n=-\infty}^{\infty} d_n e^{inw_0 t} = 2f(t+\pi) + 1$$
$$d_0 + \sum_{n=-\infty}^{\infty} d_n e^{inw_0 t} = 2\sum_{n=-\infty}^{\infty} \left(c_n e^{inw_0 t_0} e^{i\frac{nt}{2}}\right) + 1$$
$$d_0 + \sum_{n=-\infty}^{\infty} d_n e^{inw_0 t} = 1 + \sum_{n=-\infty}^{\infty} \left(2c_n e^{inw_0 t_0} e^{i\frac{nt}{2}}\right)$$

comparing both sides:

$$d_0 = 1$$
$$d_n = 2\left(\frac{1-(-1)^n}{in\pi} e^{\frac{in\pi}{2}}\right)$$
$$= 2\left(\frac{1-(-1)^n}{in\pi}(i)^n\right)$$
$$= \frac{2-2(-1)^n}{n\pi}(i)^{n-1}$$
$$d_n = 2(i)^{n-1}\left(\frac{1-(-1)^n}{n\pi}\right)$$
$$g(t) = 1 + \sum_{n=-\infty}^{\infty} \frac{2(i)^{n-1}(1-(-1)^n)}{n\pi} e^{int}$$

(vii) Plot its amplitude spectrum and phase spectrum for $-7 \leq n \leq 7$.

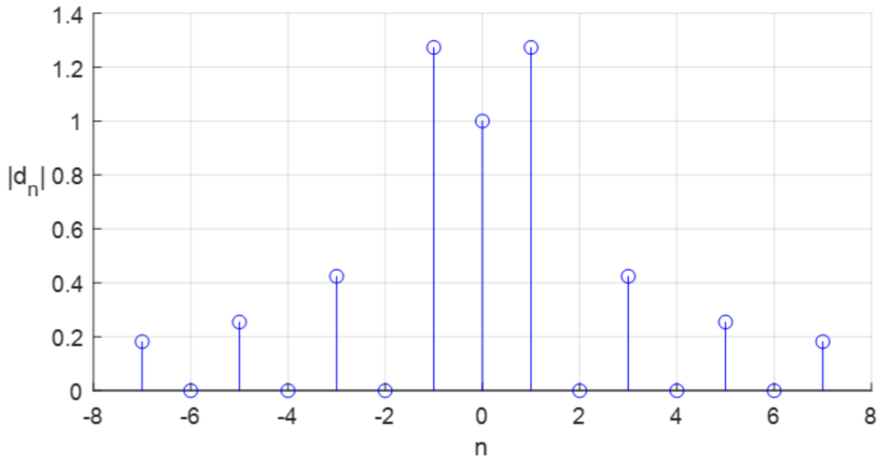


Figure 3: Amplitude Spectrum

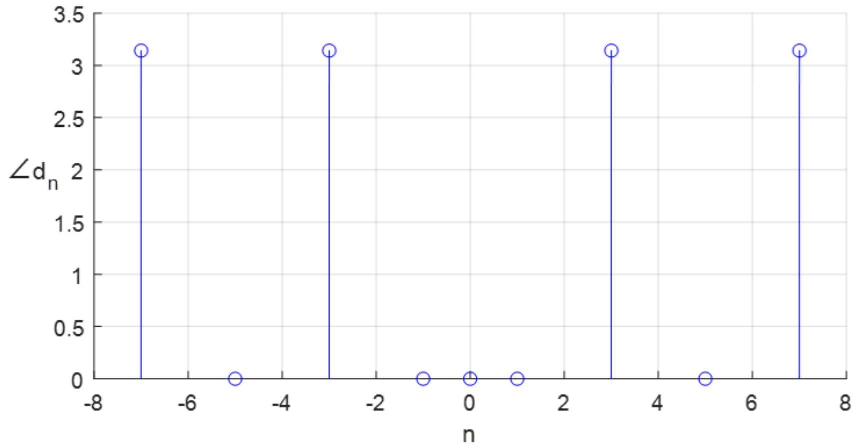


Figure 4: Phase Spectrum

(viii) Calculate the average power in $g(t)$.

$$P_{avg} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt$$

$$P_{avg} = \frac{1}{4\pi}\left(\int_{-\pi}^{\pi}(3)^2 dt + \int_{\pi}^{3\pi}(1)^2 dt\right)$$

$$P_{avg} = \frac{1}{4\pi}\left(9t\big|_{-\pi}^{\pi} + t\big|_{\pi}^{3\pi}\right)$$

$$P_{avg} = \frac{1}{4\pi}(20\pi)$$

$$P_{avg} = 5$$

(ix) Plot its power spectrum $|d_n|^2$ for $-7 \leq n \leq 7$.

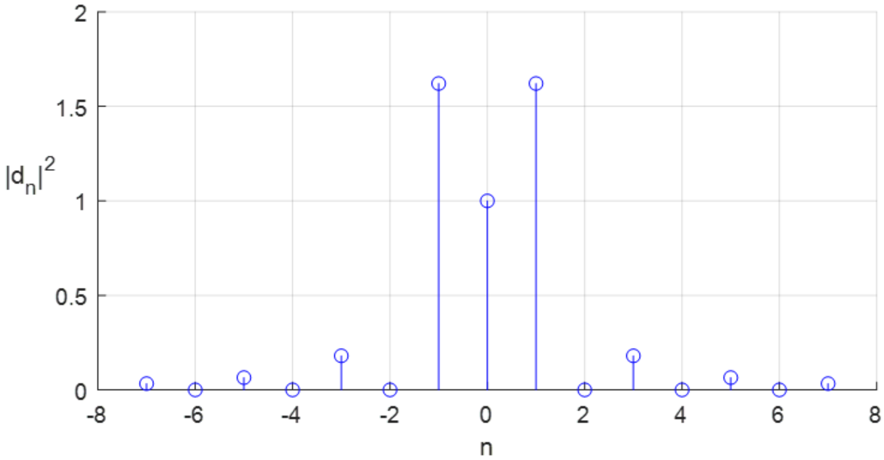
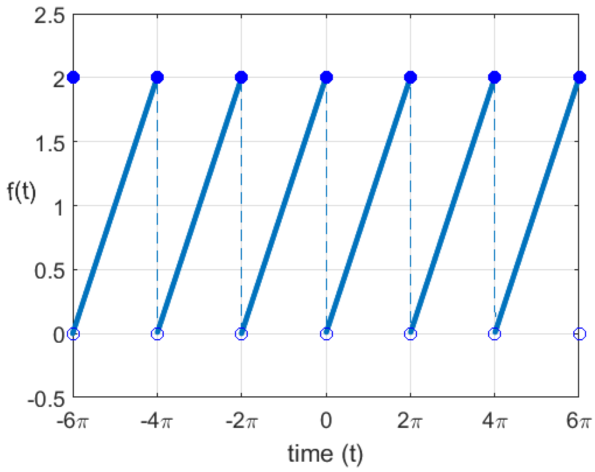


Figure 5: Power Spectrum

Problem 3

(a) Consider the periodic function shown in the following figure.



(i) Find its time period T and angular frequency ω_0.
$T = 2\pi$ and $\omega_0 = 1$

(ii) Is it an even function or an odd function?
Neither Even nor Odd

(iii) Based on (ii), determine whether its complex Fourier series coefficients c_n be real or pure imaginary? Explain.
c_n will be complex as $a_n \neq 0$ and $b_n \neq 0$

(iv) Based on (ii), determine the relationship between c_n and c_{-n}.
$c_{-n} = \bar{c_n}$

(v) Write down the equation of the function in the interval $-\frac{T}{2} < t \leq \frac{T}{2}$.

$$f(t) = \begin{cases} \frac{t}{\pi}, & 0 < t \leq \pi \\ \frac{t}{\pi} + 2, & -\pi < t \leq 0 \end{cases}$$

(vi) Now find its complex Fourier series. Clearly write the final expressions for c_0 and c_n.

$$c_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt$$

$$c_0 = \frac{1}{2\pi}\left(\int_{-\pi}^{0} \frac{t}{\pi} + 2dt + \int_{0}^{\pi} \frac{t}{\pi}dt\right) = 1$$

$$c_n = \frac{1}{2\pi}\left(\int_{-\pi}^{0} (\frac{t}{\pi} + 2)e^{-int}dt + \int_{0}^{\pi} (\frac{t}{\pi})e^{-int}dt\right)$$

$$c_n = \frac{1}{2\pi}\left(\left(\frac{te^{-int}}{-in} + \frac{e^{-int}}{n^2}\right)\Big|_{-\pi}^{0} + \left(\frac{2e^{-int}}{-in}\right)\Big|_{-\pi}^{0} + \left(\frac{te^{-int}}{-in} + \frac{e^{-int}}{n^2}\right)\Big|_{0}^{\pi}\right)$$

$$c_n = \left(\frac{1}{2n^2\pi^2} - \frac{(-1)^n}{2in\pi} - \frac{(-1)^n}{2n^2\pi^2} - \frac{1}{in\pi} + \frac{(-1)^n}{in\pi} - \frac{(-1)^n}{2ni\pi} - \frac{1}{2n^2\pi^2} + \frac{(-1)^n}{2n^2\pi^2}\right)$$

$$c_n = \frac{i}{n\pi}$$

$$f(t) = 1 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{i}{n\pi}e^{int}$$

(vii) Using your answer to (vi), evaluate the Fourier series coefficients a_n and b_n.

$$c_n = \frac{a_n - ib_n}{2}$$

$$Case1(n \neq 0)$$

$$\frac{i}{n\pi} = \frac{a_n}{2} - \frac{ib_n}{2}$$

$$\text{Comparing}$$

$$a_n = 0$$

$$b_n = 2ic_n$$

$$b_n = 2i\left(\frac{i}{n\pi}\right)$$

$$b_n = -\frac{2}{n\pi}$$

$$Case2(n = 0)$$

$$c_o = \frac{a_o}{2}$$

$$a_o = 2$$

(viii) Plot its amplitude spectrum and phase spectrum for $-7 \leq n \leq 7$.

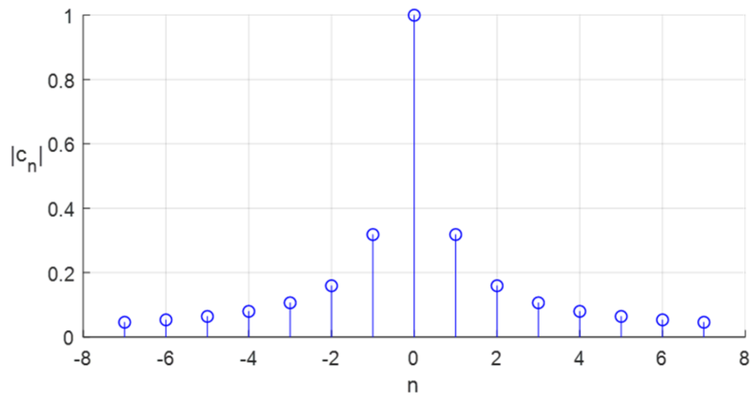


Figure 6: Amplitude Spectrum

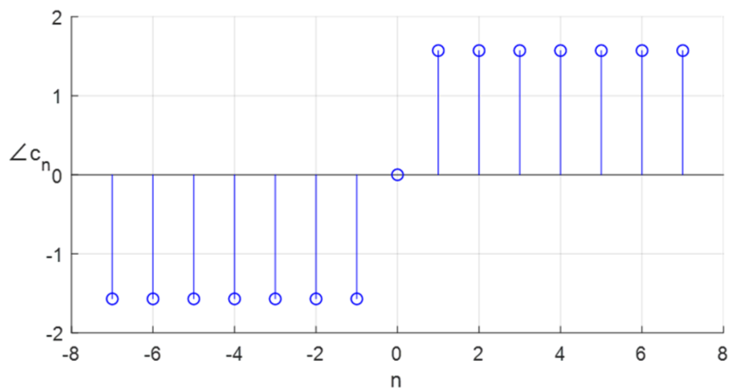
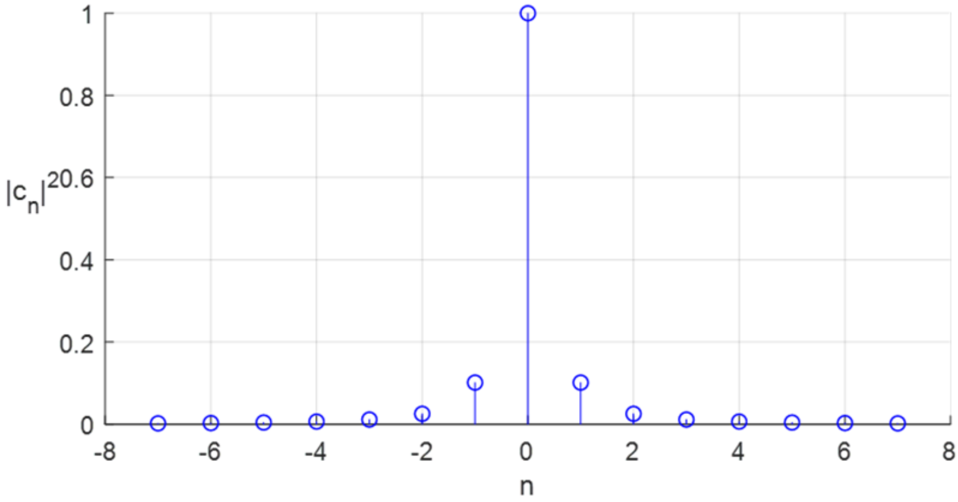


Figure 7: Phase Spectrum

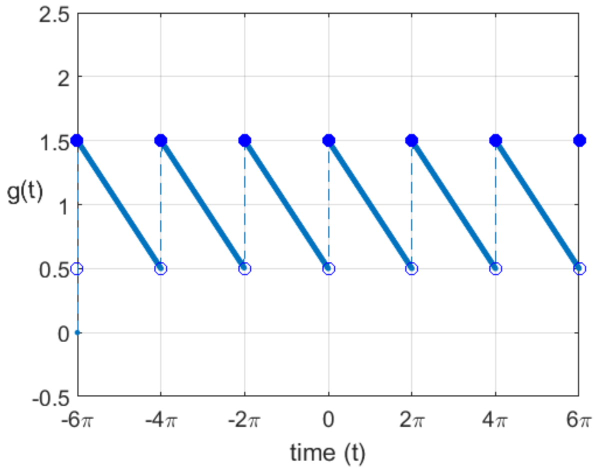
(ix) Calculate the average power in $f(t)$.

$$P_{avg} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt$$
$$P_{avg} = \frac{1}{2\pi}\left(\int_{-\pi}^{0}\left(\frac{t}{\pi}+2\right)^2 dt + \int_{0}^{\pi}\left(\frac{t}{\pi}\right)^2 dt\right)$$
$$P_{avg} = \frac{1}{2\pi}\left(2\pi + \frac{\pi}{3}\right) + \frac{1}{2\pi}\left(\frac{\pi}{3}\right)$$
$$P_{avg} = 1 + \frac{1}{6} + \frac{1}{6}$$
$$P_{avg} = \frac{4}{3}$$

(x) Plot its power spectrum $|c_n|^2$ for $-7 \leq n \leq 7$.



(b) Now consider the periodic function $g(t)$ shown in the following figure. This function is somehow related to $f(t)$ from part (a).



(i) Find its time period T and angular frequency ω_0.
$T = 2\pi$ and $\omega_0 = 1$

(ii) Is it an even function or an odd function?
Neither Even nor Odd

(iii) Based on (ii), determine whether its complex Fourier series coefficients d_n be real or pure imaginary? Explain.
d_n will be Complex.

(iv) Based on (ii), determine the relationship between d_n and d_{-n}.
$d_{-n} = \bar{d_n}$

(v) Determine the relationship between $f(t)$ and $g(t)$ and write down an expression for $g(t)$ in terms of $f(t)$. [Hint: $g(t) = af(t - t_0) + b$. Determine a, b and ϕ.]
$g(t) = \frac{1}{2}f(-t) + \frac{1}{2}$

(vi) Now using the Fourier series coefficient of $f(t)$, find the complex Fourier series of $g(t)$. Clearly write the final expressions for d_0 and d_n. [Hint: Use properties of Fourier series.]

$$c_n = \frac{i}{n\pi}$$

Using time reversal property:

$$\begin{aligned} f(t) &\rightarrow c_n \\ f(-t) &\rightarrow c_{-n} \end{aligned}$$

And linearity property:

$$\begin{aligned} f(t) &\rightarrow c_n \\ \frac{1}{2}f(t) &\rightarrow \frac{1}{2}(c_n) \end{aligned}$$

$$g(t) = d_0 + \sum_{n=-\infty}^{\infty} d_n e^{int} = \frac{1}{2}\left(1 + \sum_{n=-\infty}^{\infty} c_n e^{-int}\right) + \frac{1}{2}$$

$$g(t) = d_0 + \sum_{n=-\infty}^{\infty} d_n e^{int} = \frac{1}{2}f(-t) + \frac{1}{2}$$

$$d_0 + \sum_{n=-\infty}^{\infty} d_n e^{int} = \frac{1}{2}\left(1 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{-i}{n\pi} e^{-int}\right) + \frac{1}{2}$$

$$d_0 + \sum_{n=-\infty}^{\infty} d_n e^{int} = 1 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{-i}{2n\pi} e^{-int}$$

comparing both sides:

$$d_0 = 1$$

$$d_n = -\frac{i}{2n\pi}$$

$$g(t) = 1 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{-i}{2n\pi} e^{-int}$$

(vii) Plot its amplitude spectrum and phase spectrum for $-7 \leq n \leq 7$.

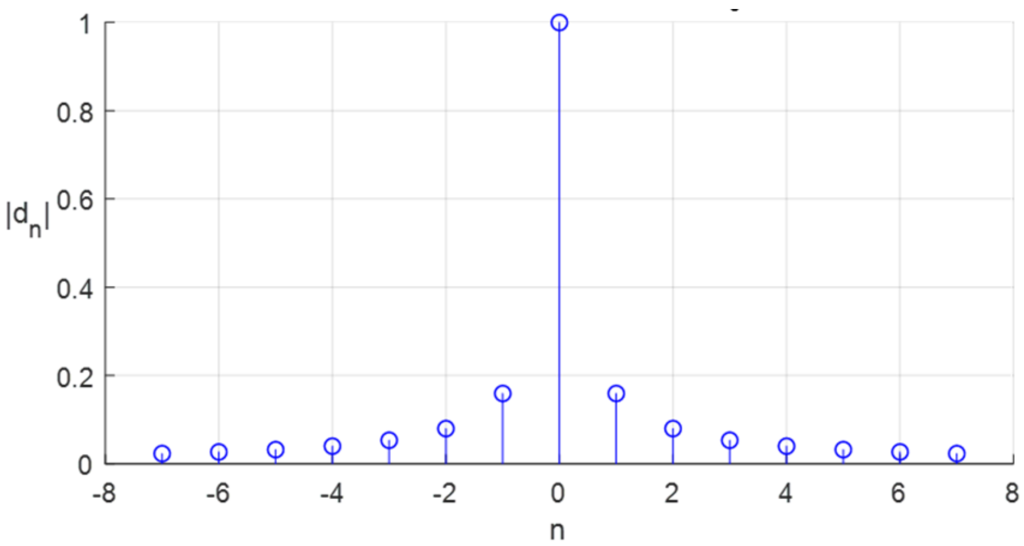


Figure 8: Amplitude Spectrum

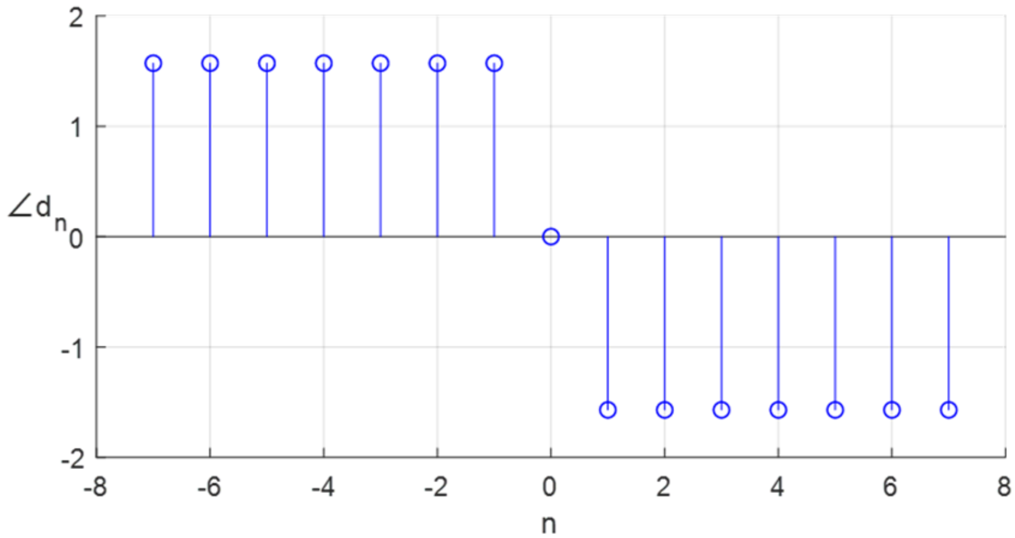


Figure 9: Phase Spectrum

(viii) Calculate the average power in $g(t)$.

$$P_{avg} = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt$$

$$P_{avg} = \frac{1}{2\pi}\left(\int_{-\pi}^{0}\left(\frac{-t}{\pi}\right)^2 dt + \int_{0}^{\pi}\left(\frac{t}{2\pi}+1.5\right)^2 dt\right)$$

$$P_{avg} = \frac{1}{2\pi}\left(\frac{\pi}{3}+\frac{\pi}{12}+(1.5)^2\pi+\frac{1.5\pi}{2}\right)$$

$$P_{avg} = \frac{4+1+27+9}{24}$$

$$P_{avg} = \frac{41}{21}$$

(ix) Plot its power spectrum $|d_n|^2$ for $-7 \leq n \leq 7$.

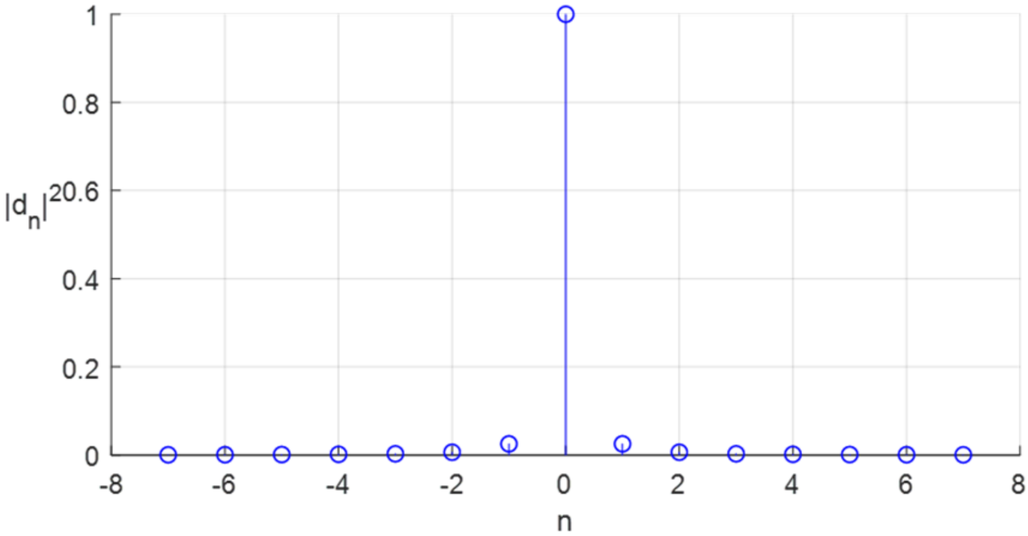


Figure 10: Power Spectrum
