MT240: Complex Variables and Transforms

Homework 3 Solution

Due 8 am, Fri Mar 22 *Spring 2019*

Tips to avoid plagiarism

- Do not copy the solutions of your classmates.
- Your are encouraged to discuss the problems with your classmates in whatever way you like but make sure to REPRODUCE YOUR OWN SOLUTIONS in what you submit for grading.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1

Evaluate the following, expressing all your answer in the form $x + iy$.

(a) $e^{2-i\frac{\pi}{4}}$

(b) $\cos(1+i)$

(c) $\sin(-i)$

(d) $\log(-1+i)$

(e) $\log(i)$

(f) $\text{Log}(-2)$

(g) $\text{Log}(1-i\sqrt{3})$

(h) $2^{2-i\frac{\pi}{4}}$

(i) i^{1+i}

(j) $(1+i)^{1-i}$

Solution

(a.) $e^2 e^{\frac{-\pi}{4}}$

$e^2(\cos(\frac{\pi}{4}) - (i\sin(\frac{\pi}{4}))$

$e^2(\cos(\frac{\pi}{4}) - ie^2(\sin(\frac{\pi}{4})$

(b.) $\cos(1+i)$

$\cos(z) = (\frac{e^{iz}+e^{-iz}}{2})$

$\cos(1+i) = (\frac{e^{i}e^{-1}+e^{-i}e^{1}}{2})$

$= (\frac{(e^{-1}(\cos(1)-(i\sin(1))+(e^{1}(\cos(1)-(i\sin(1))}{2})$

$= (\frac{\cos(1)(e^{-1}+e^{1})}{2}) + i(\frac{\sin(1)(e^{-1}-e^{1})}{2})$

(c.) $sin(-i)$

$\sin(z) = (\frac{e^{iz}-e^{-iz}}{2i})$

$\sin(i) = (\frac{e^{i}-e^{-1}}{2i})$

$= (\frac{e^{1}}{2i}) - (\frac{e^{-1}}{2i})$

$= 0 + i(\frac{(-e^{1}+e^{-1})}{2})$

(d.) $log(-1+i)$
$ln|-1+i| + iarg(-1+i)$
$ln(\sqrt{2}) + i((\frac{3\pi}{4}) + 2n(\pi))$
for $n = 0, \pm 1, \pm 2....$

(e.) $log(i)$
$ln|i| + iarg(i)$
$ln(1) + i((\frac{\pi}{2}) + 2n(\pi))$
for $n = 0, \pm 1, \pm 2....$

(f.) $Log(-2)$
$ln|-2| + iArg(-2)$
$ln(2) + i(\pi)$

(g.) $Log(1 - i\sqrt{3})$
$ln|1 - i\sqrt{3}| + iArg(1 - i\sqrt{3})$
$ln(2) - i(\frac{\pi}{3})$

(h.) $2^{2-i\frac{\pi}{4}}$
$2^2.2^{\frac{-i\pi}{4}}$
$4.e^{(\frac{-i\pi}{4})^L}og(2)$
$4.e^{(\frac{-i\pi}{4})^(}ln2 + i\pi)$
$4.e^{(\frac{-\pi}{4})}.e^{(\frac{-i\pi}{4}} + ln2))$
$4.e^{(\frac{-\pi}{4})}.(\cos(ln2\frac{\pi}{4})) - i(\sin(ln2\frac{\pi}{4})$

(i.) i^{1+i}
$(i)(i)^i$
$i(e^{i\text{Log}i})$
$ie^{i(\ln 1 + i\frac{\pi}{2})}$
$ie^{i(i\frac{\pi}{2})}$
$ie^{-\frac{\pi}{2}}$

(j.) $(1+i)^{1-i}$
$(1+i)(1+i)^{-i}$
$(1+i)e^{\frac{\pi}{4}}e^{-i\ln\sqrt{2}}$
$(1+i)e^{\frac{\pi}{4}}(\cos(\ln\sqrt{2}) - i\sin(\ln\sqrt{2}))$
$e^{\frac{\pi}{4}}\left(\cos(\ln\sqrt{2}) + \sin(\ln\sqrt{2})\right) + ie^{\frac{\pi}{4}}\left(\cos(\ln\sqrt{2}) - \sin(\ln\sqrt{2})\right)$

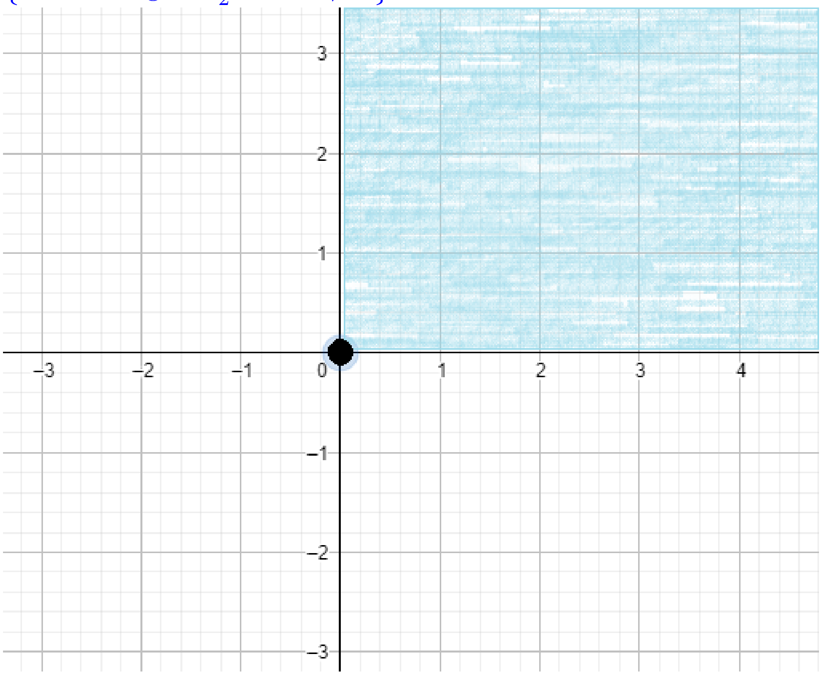
Problem 2

Consider the function $f(z) = \frac{1}{z^4}$.

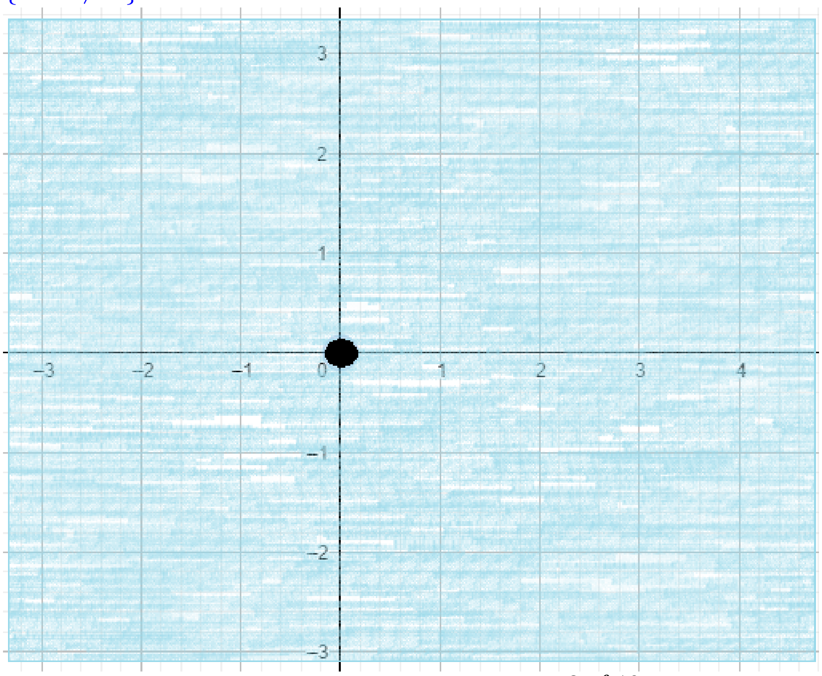
(a) Find all the points where $f'(z) = 0$.
$f'(z) = \frac{-4}{z^5}$
"$f'(z)$ is not zero anywhere in the complex plan at any value of z"

(b) Show that there is a domain on which the function is one-to-one.
$\{z : z \neq 0\}$

(c) Specify a domain in which the function in one-to-one and sketch it in the z-plane.
Specifying domain in 1st quadrant
$\{z : 0 < \text{Arg}z < \frac{\pi}{2} \quad \wedge \quad z \neq 0\}$



(d) If possible, define the corresponding range of $f(z)$ in the set notation and sketch it in the w-plane.
$\{z : z \neq 0\}$



(e) If possible, find its inverse function $f^{-1}(z)$.

$w = \frac{1}{z^4}$
$z = \frac{1}{w^{\frac{1}{4}}}$
$f^{-1}(z) = \frac{1}{z^{\frac{1}{4}}}$
$f^{-1}(w) = \frac{1}{w^{\frac{1}{4}}}$

Problem 3

Find a Möbius transformation which fulfills each of the following conditions and express in the form

$$f(z) = \frac{az + b}{cz + d}, \quad \text{for } a, b, c, d \in \mathbb{C}.$$

(a)
- $f(1) = 0$
- $f(0) = 1$
- $f(-1) = \infty$

(b)
- $f(0) = 0$
- $f(1) = 1 + i$
- $f(2i) = \infty$

(c)
- $f(1) = 0$
- $f(\infty) = 1$
- $f(-1) = \infty$

(d)
- $f(0) = -i$
- $f(1) = \infty$
- $f(\infty) = 1$

Solution

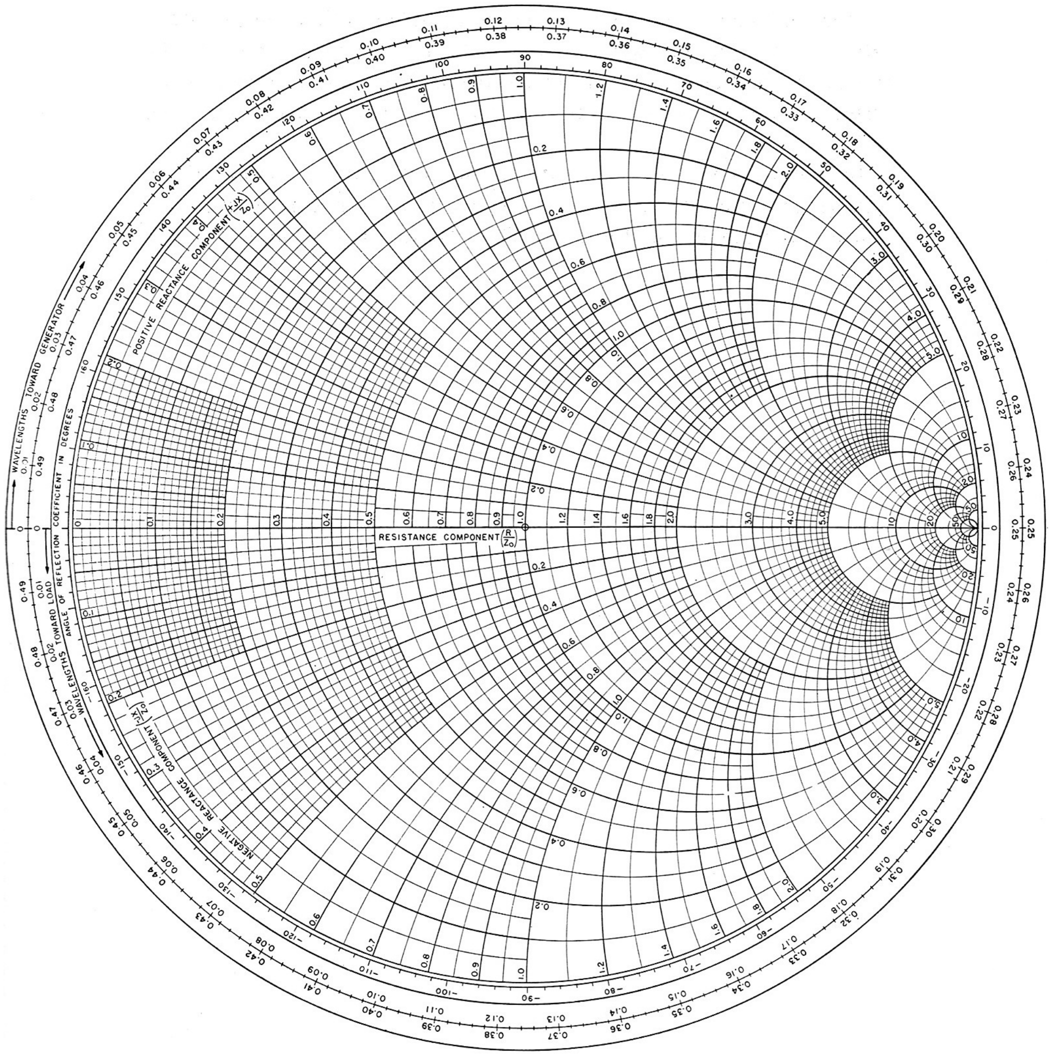
We use
$f(z) = \frac{z-z_1}{z-z_3}\left(\frac{z_2-z_3}{z_2-z_1}\right)$

(a) $f(z) = -(\frac{z-1}{z+1})$

(b) $f(z) = (\frac{(z)(1-2i)(1+i)}{z-2i})$

(c) $f(z) = (\frac{z-1}{z+1})$

(d) In this Part we use
$f(w) = \frac{w-w_1}{w-w_3}\left(\frac{w_2-w_3}{w_2-w_1}\right)$
$f(w) = \frac{w-i}{w-1}\left(\frac{\infty-1}{\infty+i}\right)$
$f(w) = \frac{w+i}{w-1}$
$f^{-1}(w) = \frac{w+i}{w-1} = z$
$\frac{w+i}{w-1} = z$
$w + i = zw - Z$
$w = \frac{z-i}{1-z}$
Now Plug in the z values in above expression and you will get respective w values.



Smith Chart

Problem 4

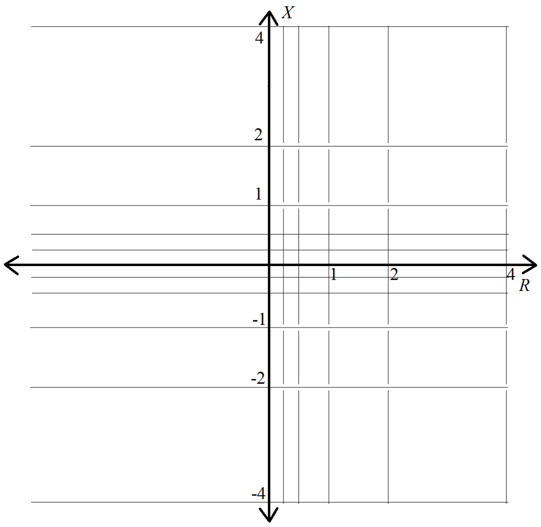
The Smith chart, invented by Phillip H. Smith (1905-1987), is a graphical aid designed for electrical and electronics engineers specializing in radio frequency (RF) engineering to assist in solving problems with transmission lines and matching circuits. [Wikipedia]

A Smith chart is a circular plot with a lot of interlaced circles on it. When used correctly, matching impedances, with apparent complicated structures, can be made without any computation. The Smith chart provides a more compact graphical description, displaying the entire range of impedance within the unit circle.

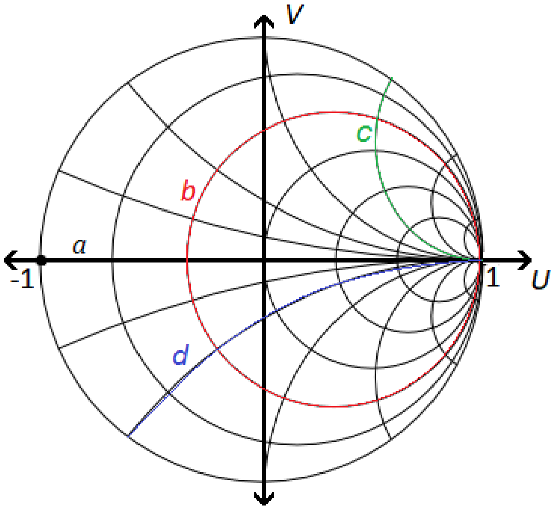
The impedance Z of an electrical circuit oscillating at a frequency ω is a complex number, denoted $Z = R + iX$, which characterizes the voltage-current relationship of the circuit. In practice R (resistance) can take any value from 0 to ∞ and X (reactance) can take any value from $-\infty$ to ∞. Through a Mobius transformation, the impedance Z is mapped to the W-plane and depicted as the point

$$W = \frac{Z-1}{Z+1},$$

where W is known as the reflection coefficient corresponding to Z.



Complex impedance Z-plane



Smith chart W-plane

Figure on the left shows various vertical and horizontal lines of constant values of R and X in the Z-plane. When each of these lines is transformed through the Mobius transformation given above, it maps to either a circle or a circular arc drawn on the Smith chart in figure on the right.

The 6 vertical lines in the Z-plane are $R = 0, R = \frac{1}{4}, R = \frac{1}{2}, R = 1, R = 2, R = 4$.

The 11 horizontal lines in the Z-plane are $X = 0, X = \pm\frac{1}{4}, X = \pm\frac{1}{2}, X = \pm 1, X = \pm 2, X = \pm 4$.

(i) Find out the four lines that map to each of the following four curves on the Smith chart.

(a) The real axis, drawn in black and marked by the letter a.

(b) The circle, drawn in red and marked by the letter b.

(c) The ciruclar arc, drawn in green and marked by the letter c.

(d) The ciruclar arc, drawn in blue and marked by the letter d.

(ii) Find out the three values of Z, in the form $R + iX$, corresponding to the three points where b intersects a, c and d.

[Hint: A straight-forward approach is to start mapping each of the lines drawn in the Z-plane and figure out which ones are mapped to a, b, c and d. But a smarter approach is to find an inverse transformation (function) that maps W to Z and then work backwards.]

Solution

$$W = 1 - \frac{2}{Z+1}$$
$$Z = -1 - \frac{2}{W-1}$$

- We can observe that the R-axis in the Z-plane maps to the U-axis in the W plane
- The horizontal lines above the R-axis in Z-plane map to the arcs above the U-axis in W-plane
- The horizontal lines below the R-axis in Z-plane map to the arcs below the U-axis in W-plane

(i) Find out the four lines that map to each of the following four curves on the Smith chart.

(a) The real axis, drawn in black and marked by the letter a.
$X = 0$

(b) The circle, drawn in red and marked by the letter b.
$R = \frac{1}{2}$

(c) The ciruclar arc, drawn in green and marked by the letter c.
$X = 2$

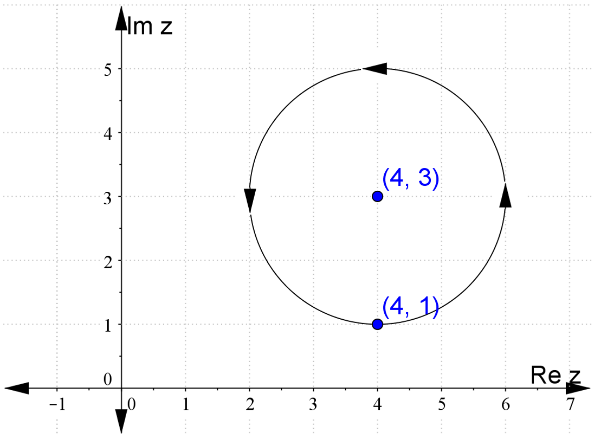
(d) The ciruclar arc, drawn in blue and marked by the letter d.
$X = -\frac{1}{2}$

(ii) Find out the three values of Z, in the form $R + iX$, corresponding to the three points where b intersects a, c and d.
$R = \frac{1}{2}$ intersects $X = 0, X = -\frac{1}{2}, X = 2$

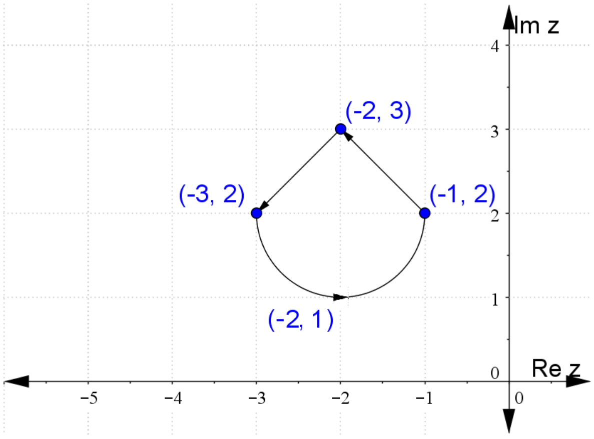
$Z = \frac{1}{2}$, $Z = \frac{1}{2} - i\frac{1}{2}$, $Z = \frac{1}{2} + i2$

Problem 5

For each of the following closed contours (piecewise smooth curves) in the complex plane,



(a)



(b)

(i) Parameterize the curve

(ii) Evaluate the integral $\oint_\gamma \frac{1}{z + 2 - 2i}\, dz$ over the curve using

(1) The parameterization you found in (i)

(2) Cauchy's integral theorem or formula

(iii) Evaluate the integral $\oint_\gamma \frac{z^2}{(z - 3 - 2i)(z + 2 - 2i)^2}\, dz$ over the curve using Cauchy's integral formula, simplifying your answer in the form $x + iy$.

Solution

(i) Parameterize the curve

(a) $\gamma_a = 2e^{it} + 4 + 3i \quad t : [0, 2\pi]$

(b) $\gamma_b = \gamma_1 + \gamma_2 + \gamma_3$
$\gamma_1 = e^{it} - 2 + 2i \quad t : [0, \pi]$

$z = x + iy$ $\qquad$ $y = -x + 1$

$\gamma_2 = t + i(-t + 1)$ $\qquad$ $x = t$

$t : [-1, -2]$

$z = x + iy$ $\qquad$ $y = x + 5$

$\gamma_3 = t + i(t + 5)$ $\qquad$ $x = t$

$t : [-2, -3]$

(ii) Evaluate the integral $\oint_\gamma \frac{1}{z + 2 - 2i}\, dz$ over the curve using

(1) The parameterization you found in (i)

For Figure (a)

$$\gamma_a = 2e^{it} + 4 + 3i \quad t : [0, 2\pi]$$
$$\gamma_a' = 2ie^{it}$$
$$\int_0^{2\pi} \frac{1}{2e^{it} + 6 + i}(2ie^{it})\, dt$$
$$= |\ln(2e^{it} + 6 + i)|_0^{2\pi}$$
$$= \ln(8 + i) - \ln(8 + i) \;=\; 0$$

For Figure (b)

$\gamma_b = \gamma_1 + \gamma_2 + \gamma_3$

$$\gamma_1 = e^{it} - 2 + 2i \quad t : [\pi, 2\pi]$$
$$\gamma_1' = ie^{it}$$
$$\int_\pi^{2\pi} \frac{1}{e^{it}}(ie^{it})\, dt$$
$$= i|t|_\pi^2\pi$$

$$\gamma_2 = t - it + i \quad t : [-1, -2]$$
$$\gamma_2' = 1 - i$$
$$\int_{-1}^{-2} \frac{1}{t - it + 2 - i}(1 - i)\, dt$$
$$= |\ln(t - it + 2 - i)|_{-1}^{-2}$$

$$\gamma_3 = t + it + 5i \quad t : [-2, -3]$$
$$\gamma_3' = 1 + i$$
$$\int_{-2}^{-3} \frac{1}{t + it + 2 + 3i}(1 + i)\, dt$$
$$= |\ln(t + it + 2 + 3i)|_{-2}^{-3}$$

$$\begin{aligned}\int_{\gamma_b} &= \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \\ &= [i\pi] + [\ln(i) - \ln(1)] + [\ln(-1) - \ln(i)] \\ &= [i\pi] + [i\frac{\pi}{2}] + [i\pi - i\frac{\pi}{2}] \\ &= 2\pi i\end{aligned}$$

(2) Cauchy's integral theorem or formula

Function: $\frac{1}{z+2-2i}$
Not analytic at: $z_0 = -2 + 2i$

(a) The point z_0 does not lie in the curve γ_a
Using Cauchy Integral Theorem we can calculate;

$$\oint_{\gamma_a} \frac{1}{z+2-2i}\, dz = 0$$

(b) The point z_0 lies in the curve γ_b
Using Cauchy's Integral Formula we can calculate;

$$\begin{aligned}f(z_0) &= \frac{1}{2\pi i} \oint_{\gamma_b} \frac{1}{z+2-2i}\, dz \\ f(z) &= 1 \\ f(-2+2i) &= 1 \\ \oint_{\gamma_b} \frac{1}{z+2-2i}\, dz &= 2\pi i\end{aligned}$$

(iii) Evaluate the integral $\oint_{\gamma} \frac{z^2}{(z-3-2i)(z+2-2i)^2}\, dz$ over the curve using Cauchy's integral formula, simplifying your answer in the form $x + iy$.

Function: $\frac{z^2}{(z-3-2i)(z+2-2i)^2}$
Analytic except at$z = 3 + 2i$ and $z = -2 + 2i$

(a) Singularity $3 + 2i$ lies inside the curve
Using Cauchy's Integral Formula, we evaluate;

$$\begin{aligned}f(z_0) &= \frac{1}{2\pi i} \oint_{\gamma_a} \frac{z^2}{(z-3-2i)(z+2-2i)^2}\, dz \\ f(z) &= \frac{z^2}{(z+2-2i)^2} \\ f(3+2i) &= \frac{(3+2i)^2}{(3+2i+2-2i)^2} = \frac{5}{25} + i\frac{12}{25} \\ \frac{5+12i}{25} &= \frac{1}{2\pi i} \oint_{\gamma_a} \frac{z^2}{(z-3-2i)(z+2-2i)^2}\, dz \\ \oint_{\gamma_a} \frac{z^2}{(z-3-2i)(z+2-2i)^2}\, dz &= -\frac{4\pi}{25} + i\frac{10\pi}{25}\end{aligned}$$

(b) Singularity $-2+2i$ lies inside the curve

$$f'(z_0) = \frac{1}{2\pi i}\oint_{\gamma_b} \frac{z^2}{(z-3-2i)(z+2-2i)^2}\,dz$$

$$f(z) = \frac{z^2}{z-3-2i}$$

$$f'(z) = \frac{z^2-6z-4iz}{(z-3-2i)^2}$$

$$f(-2+2i) = \frac{(-2+2i)^2-6(-2+2i)-4i(-2+2i)}{(-2+2i-3+2i)^2} = \frac{20-12i}{25}$$

$$\frac{20-12i}{25} = \frac{1}{2\pi i}\oint_{\gamma_b} \frac{z^2}{(z-3-2i)(z+2-2i)^2}\,dz$$

$$\oint_{\gamma_b} \frac{z^2}{(z-3-2i)(z+2-2i)^2}\,dz = \frac{24\pi}{25} + i\frac{40\pi}{25}$$