MT240: Complex Variables and Transforms

Homework 2 Solution

Due 8 am, Fri Mar 8 *Spring 2019*

Tips to avoid a case of plagiarism:

- You must NOT look at the solutions of your classmates.
- Your are encouraged to discuss with your classmates, but restrict the discussions to using just your mouths and facial expressions.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1

Sketch the image of each of the following functions defined on the given domains, and write down the corresponding range in set notation.

Tip: First map the boundary of domain to the u-v plane. Then choose an interior point of the domain, map it to u-v plane and see on what side of the boundary does it get mapped in the u-v plane.

(a) $f(z) = z + 1 + 2i,$ for $\{z : |z| < 1\}$

(b) $f(z) = 2z,$ for $\{z : |z| < 1\}$

(c) $f(z) = \left(1 - i\sqrt{3}\right) z,$ for $\{z : |z| < 1 \quad \wedge \quad \text{Im } z > 0\}$ [Hint: Convert $1 - i\sqrt{3}$ to the polar form]

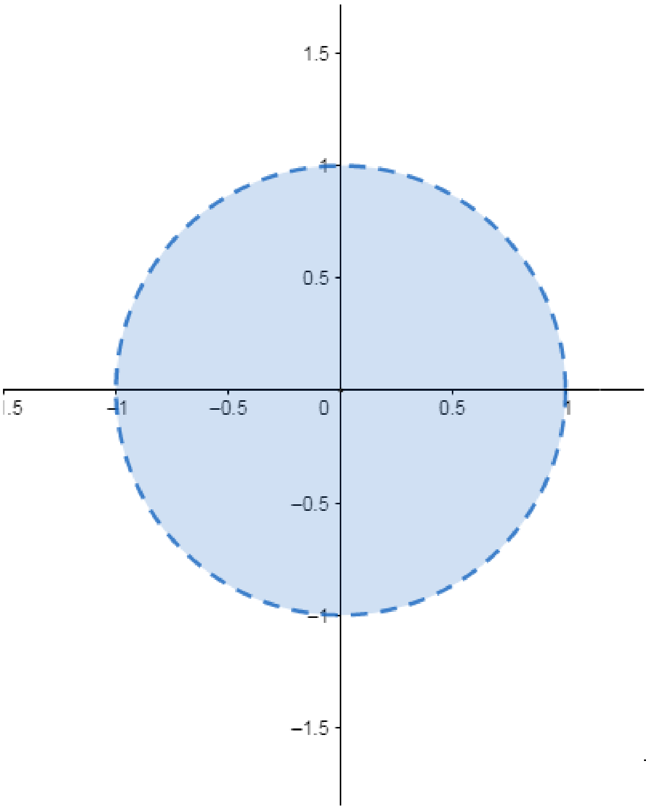
(d) $f(z) = \left(1 - i\sqrt{3}\right) z + 1 + 2i,$ for $\{z : |z| < 1 \quad \wedge \quad \text{Im } z > 0\}$

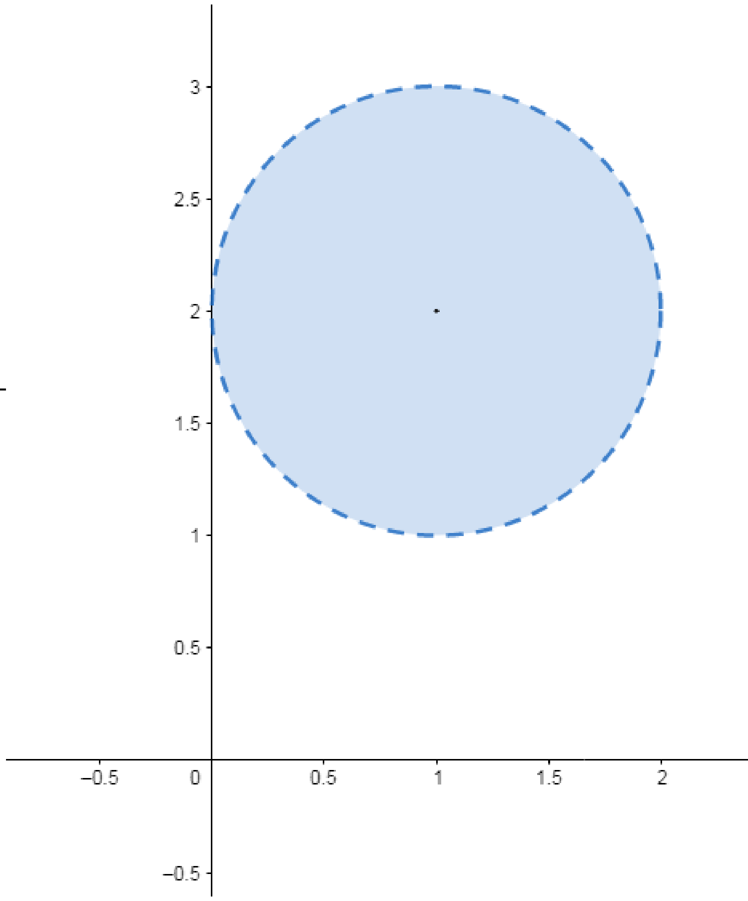
(e) $f(z) = \dfrac{1}{z},$ for $\left\{z : \frac{1}{2} < |z| < 1\right\}$ [Hint: Polar form]

(f) $f(z) = z^3,$ for $\{z : \text{Re } z = -\text{Im } z\}$

Solution

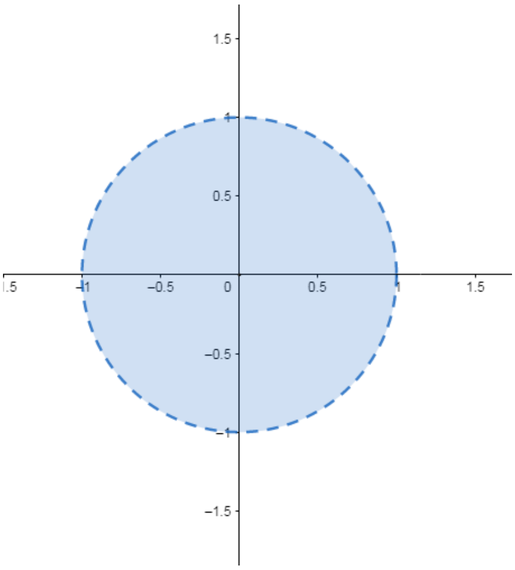
(a) $f(z) = z + 1 + 2i,$ defined on $\{z : |z| < 1\}$

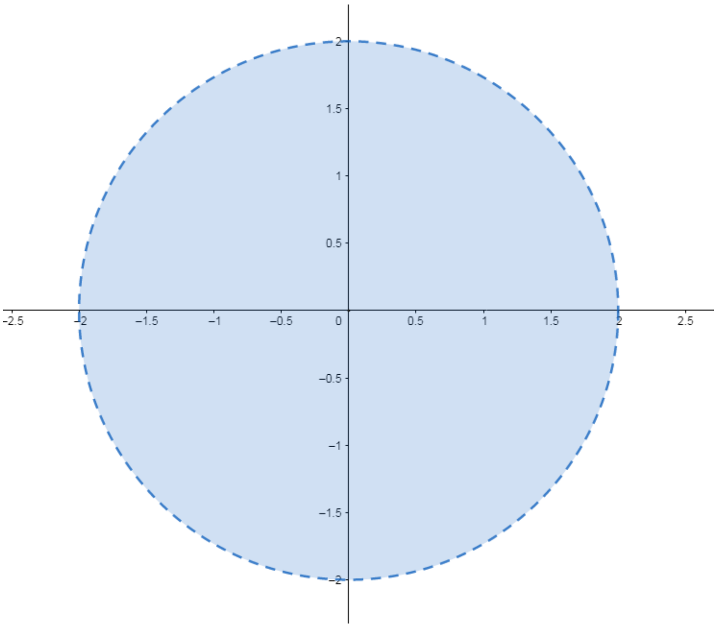




Range: $\{z : |z - (1 + 2i)| < 1\}$

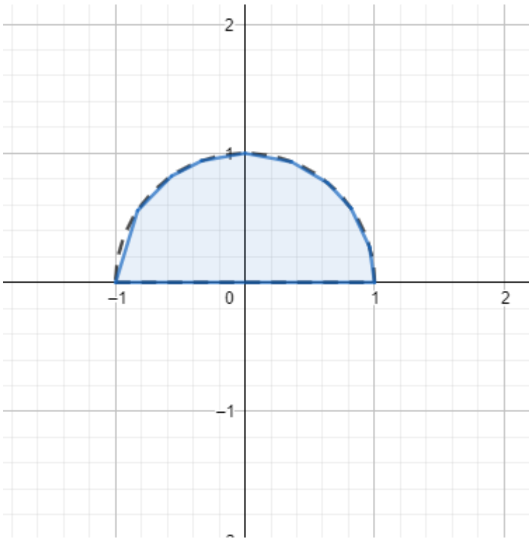
(b) $f(z) = 2z,$ defined on $\{z : |z| < 1\}$

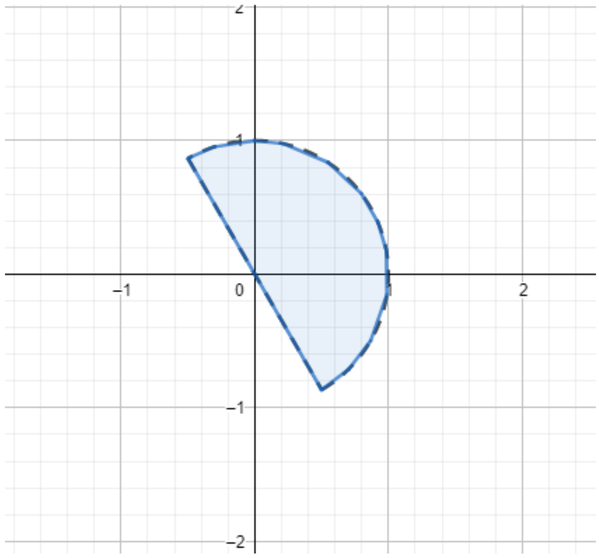




Range: $\{z : |z| < 2\}$

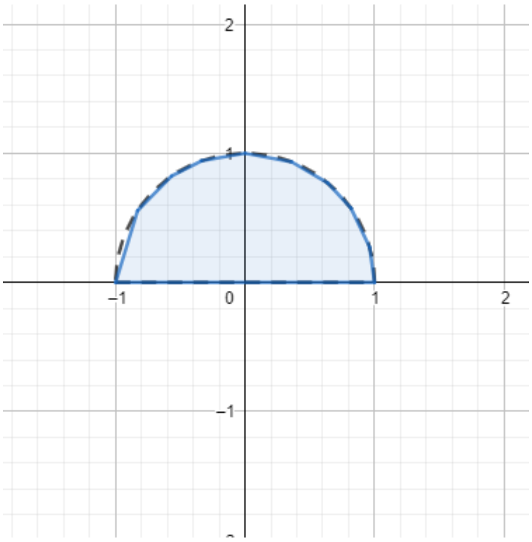
(c) $f(z) = \left(1 - i\sqrt{3}\right) z, \text{ for } \{z : |z| < 1 \quad \wedge \quad \text{Im } z > 0\}$ [Hint: Convert $1 - i\sqrt{3}$ to the polar form]

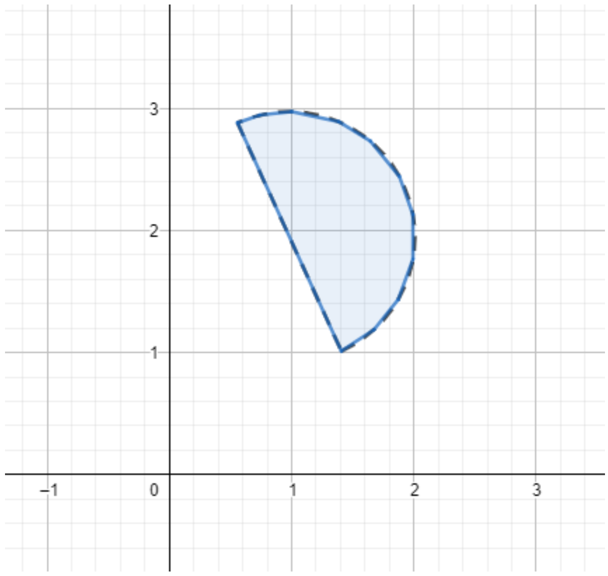




Range: $\{z : |z| < 1 \quad \wedge \quad \frac{2\pi}{3} < \text{Arg z} \ < \frac{-\pi}{3}\}$

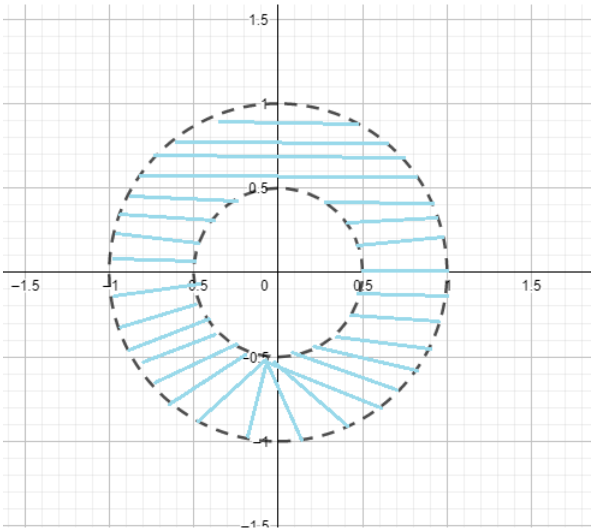
(d) $f(z) = \left(1 - i\sqrt{3}\right) z + 1 + 2i, \text{ for } \{z : |z| < 1 \quad \wedge \quad \text{Im } z > 0\}$

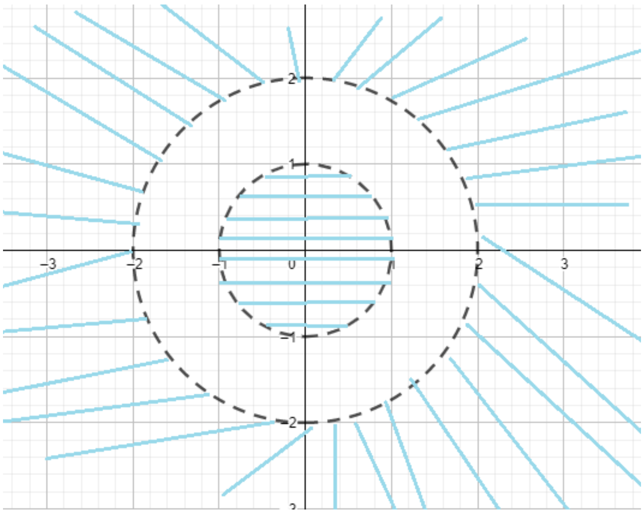




Range: $\{z : |z - (1 + 2i)| < 1 \quad \wedge \quad \frac{2\pi}{3} < \text{Arg z} \ < \frac{-\pi}{3}\}$

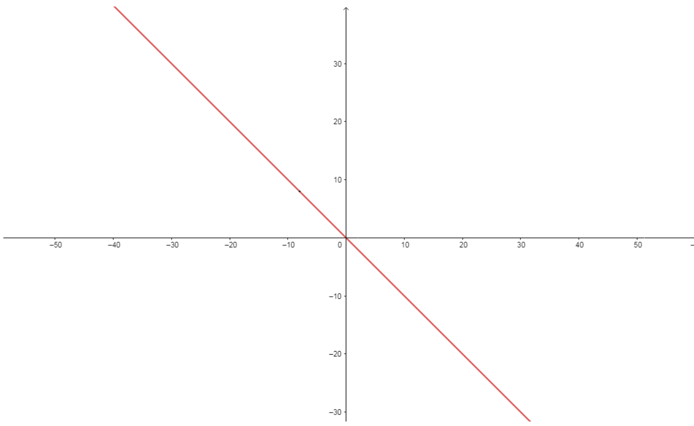
(e) $f(z) = \dfrac{1}{z}$, for $\left\{z : \frac{1}{2} < |z| < 1\right\}$ [Hint: Polar form]

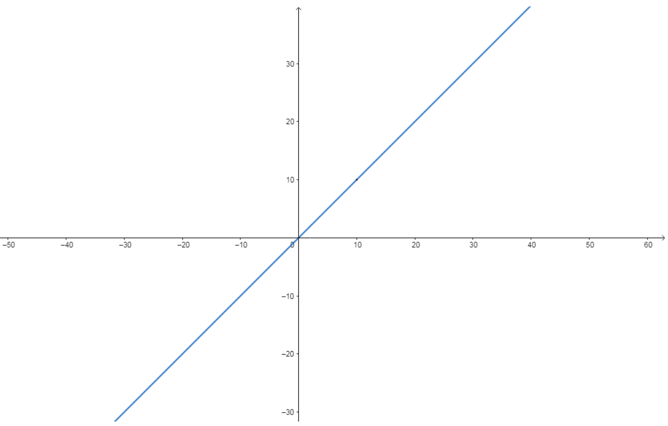




Range: $\{z : 2 < |z| < 1\}$

(f) $f(z) = z^3$, for $\{z : \text{Re } z = -\text{Im } z\}$





Range: $\{z : Rez = Imz\}$

Problem 2

Evaluate the following limits and write your answer in the form $x + iy$. If the limit exists, determine whether the function is continuous or not at this point.

In case the limit does not exist, state the reason.

(a) $\lim_{z \to 4i} \frac{z}{z^2 + 16}$ (b) $\lim_{z \to i} \frac{z^2 + 1}{z^4 - 1}$ (c) $\lim_{z \to 2i} \frac{z^2 + 4}{z}$ (d) $\lim_{z \to -1} \operatorname{Arg} z$

Solution

For a function $f(z)$ to be continuous

(i) $\lim_{z \to z_0}$ should exist

(ii) $f(z_0) = w_0$

(iii) $\lim_{z \to z_0} = w_0$

(a) $\lim_{z \to 4i} \frac{z}{z^2 + 16}$
$\lim_{z \to 4i} \frac{z}{(z + 4i)(z - 4i)} = \text{undefined}(\infty)$
Limit undefined, so not continuous

(b) $\lim_{z \to i} \frac{z^2 + 1}{z^4 - 1}$
$\lim_{z \to i} \frac{z^2 + 1}{(z^2 + 1)(z^2 - 1)}$
$\lim_{z \to i} \frac{1}{z^2 - 1} = -\frac{1}{2}$
$f(i) = \mathit{undefined}$, while $\lim_{z \to i} \frac{z^2 + 1}{z^4 - 1} = -\frac{1}{2}$
Not equal hence not continuous

(c) $\lim_{z \to 2i} \frac{z^2 + 4}{z}$
$\lim_{z \to 2i} \frac{(z + 2i)(z - 2i)}{z} = 0$
$f(2i) = 0$
Limit exists and is equal to 0. The function is continuous

(d) $\lim_{z \to -1} \operatorname{Arg} z$
Approaching from above the limit $\lim_{z \to -1} \operatorname{Arg} z \to \pi$
Approaching from below the limit $\lim_{z \to -1} \operatorname{Arg} z \to -\pi$
The limits from above and below are not equal, hence the limit $\lim_{z \to -1} \operatorname{Arg} z$ doesnot exist
Function is not continuous as the limit does not exist

Problem 3

Find a complex function that maps the lower half-plane to the 2nd quadrant.

[Hint: First find a complex function that maps the lower half-plane to the 4th quadrant. Then think about how to map the 4th quadrant to the 2nd quadrant.]

Solution

As we know
$z = re^{i\theta}$

A function that maps the lower half plane to the Fourth Quadrant, reduces the angle of the complex number by half ($\frac{\theta}{2}$)

$w = z^{\frac{1}{2}} = \pm\sqrt{z} = z^{\frac{1}{2}} = r^{\frac{1}{2}}e^{i\frac{\theta}{2}}$

A reflection along the $y = -x$ line shifts the number in 4^{th} quadrant to the 2^{nd} quadrant. This is achieved by choosing the result with negative sign $-\sqrt{z}$

Hence the desired function that maps the upper half plane to the third quadrant is: $-\sqrt{z}$ OR $(-z^{\frac{1}{2}})$

Problem 4

For each of the following functions

(a) $f(z) = xy + iy$

(b) $f(z) = iz^2$

(c) $f(z) = 2xy + i(x^2 - y^2)$

(d) $f(z) = e^{iz}$

(e) $f(z) = e^x e^{iy}$

(f) $f(z) = \ln r + i\theta$

(g) $f(z) = z + \dfrac{1}{z}$ [Hint: Polar]

(h) $f(z) = x^3 + i(1 - y)^3$

(i) Find the set in the complex plane on which the function is analytic.

(ii) In case a function is analytic on some set, find $f'(z)$ in that set.

(iii) Determine the singular points of the function.

Solution

(a) $f(z) = xy + iy$

(i) $u = xy$ $\quad v = y$

$u_x = y$ $\quad v_y = 1$

$u_y = x$ $\quad v_x = 0$

$u_x = v_y \ , \ u_y = -v_x$

$y = 1 \ , \ x = 0$

u_x, u_y, v_x, v_y are continuous So, function is Analytic at $z = i$

(ii) $f'(z) = f_x = y$

(iii) All point are Singular except $z = i$

(b) iz^2

$-2xy + i(x^2 - y^2)$

(i) $u = -2xy$ $\quad v = x^2 - y^2$

$u_x = -2y$ $\quad v_y = -2y$

$u_y = -2x$ $\quad v_x = 2x$

u_x, u_y, v_x, v_y are continuous

But C.R equations don't hold, except at $x = 0$ and $y = 0$. The function is anlaytic only at the point $x = 0 \ , \ y = 0$

(ii) $f'(z) = -2y + i2x$

(iii) No singular point

(c) $2xy + i(x^2 - y^2)$

(i) $u = 2xy$ $\quad v = x^2 - y^2$

$u_x = 2y$ $\quad v_y = -2y$

$u_y = 2x$ $\quad v_x = 2x$

u_x, u_y, v_x, v_y are continuous

But C.R equations don't hold, except at $x = 0$ and $y = 0$. The function is anlaytic only at the point $x = 0 \ , \ y = 0$

(ii) $f'(0) = 2y + i2x = 0$

(iii) All point are Singular except $z = 0$

(d) $f(z) = e^{iz}$

(i) $u = e^{-y}\cos x$ $\quad$ $v = e^{-y}\sin y$

$u_x = -e^{-y}\sin x$ $\quad$ $v_y = -e^{-y}\cos x$

$u_y = -e^{-y}\cos x$ $\quad$ $v_x = e^{-y}\sin x$

u_x, u_y, v_x, v_y are continuous

$u_x = v_y, u_y = -v_x$ (C.R holds) for all x and y; hence the function is entire

(ii) $f'(z) = f_x = e^{-y}[-\sin x + i\cos x]$

(iii) No singular point as function is Entire

(e) $f(z) = e^x e^{iy}$

(i) $u = e^x\cos y$ $\quad$ $v = e^x\sin y$

$u_x = e^x\cos y$ $\quad$ $v_y = e^x\cos y$

$u_y = -e^x\sin y$ $\quad$ $v_x = e^x\sin y$

u_x, u_y, v_x, v_y are continuous

$u_x = v_y, u_y = -v_x$ (C.R holds) for all x and y; hence the function is entire

(ii) $f'(z) = f_x = e^x[\cos y + i\sin y]$

(iii) No singular point as function is Entire

(f) $f(z) = \ln r + i\theta$

(i) $u = ln(r)$ $\quad$ $v = \theta$

$u_r = \frac{1}{r}$ $\quad$ $v_r = 0$

$u_\theta = 0$ $\quad$ $v_\theta = 1$

$u_r, u_\theta, v_r, v_\theta$ are continuous

$ru_r = v_\theta, u_\theta = -rv_r$ (C.R holds) for all r and θ; hence the function is entire

(ii) $f'(z) = e^{-i\theta} = (\frac{e^{-i\theta}}{r})$

(iii) No singular point as function is Entire

(g) $z + \dfrac{1}{z}$

$$x + iy + \frac{1}{x+iy}\left(\frac{x-iy}{x-iy}\right)$$

$$x + \frac{x}{x^2+y^2} + i\left(y - \frac{y}{x^2+y^2}\right)$$

(i) $u = x + \dfrac{x}{x^2+y^2}$ $\quad$ $v = y - \dfrac{y}{x^2+y^2}$

$u_x = 1 + \dfrac{y^2-x^2}{(x^2+y^2)^2}$ $\quad$ $v_y = 1 + \dfrac{y^2-x^2}{(x^2+y^2)^2}$

$u_y = \dfrac{-2xy}{(x^2+y^2)^2}$ $\quad$ $v_x = \dfrac{2xy}{(x^2+y^2)^2}$

u_x, u_y, v_x, v_y are continuous except at z=0
C.R holds. The function is analytic everywhere except at $z = 0$
Range: $\{z : z \subset \mathbb{C} \wedge z \neq 0\}$

(ii) $f'(z) = 1 - \dfrac{1}{z^2}$

(iii) Singular point is at $z = 0$

(h) $f(z) = x^3 + i(1 - y)^3$

(i) $u = x^3$ $\quad v = (1 - y)^3$

$u_x = 3x^2$ $\quad v_y = -3(1 - y)^2$

$u_y =$ $\quad v_x = 0$

$u_x = v_y$, $u_y = -v_x$
$x^2 = -(1 - y)^2$, $0 = 0$
u_x, u_y, v_x, v_y are continuous So, function is Analytic at $x^2 = -(1 - y)^2$

(ii) $f'(z) = f_x = 3x^2$
$f'(z) = -if_y = 3i(1 - y^2)$

(iii) All point are Singular except $x^2 = -(1 - y)^2$

Problem 5

For each of the following functions,

(a) $v(x, y) = 2y - 3x^2y + y^3$

(b) $u(x, y) = \dfrac{y}{x^2 + y^2}$ [Hint: Polar]

(c) $u(x, y) = xy - x + y$

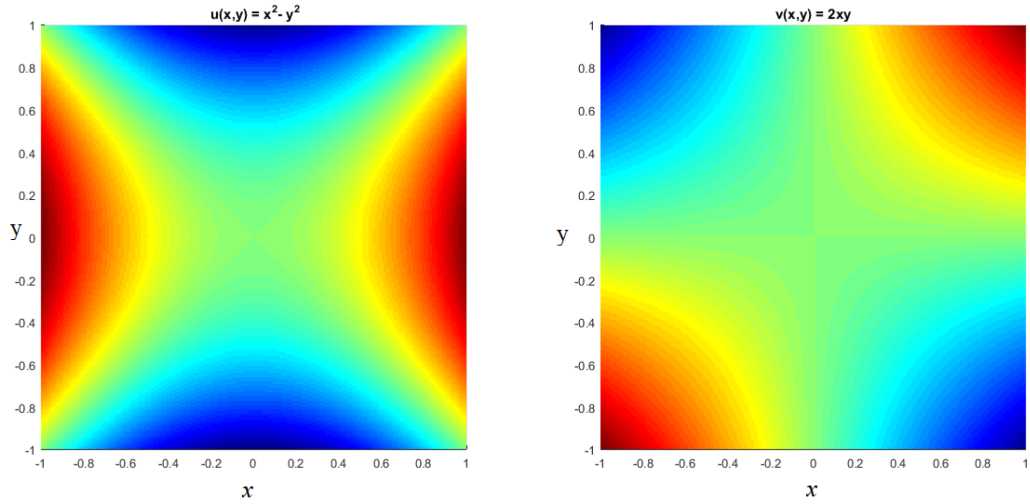
(d) $v(x, y) = x^2 - y^2 + 2y$

(i) Determine whether the function is harmonic.

(ii) If it is harmonic, find its harmonic conjugate function.

(iii) Form an analytic function $f(z) = u(x, y)+iv(x, y)$ from the harmonic function and its harmonic conjugate.

(iv) Each of the functions $u(x, y)$ and $v(x, y)$ can be plotted in MATLAB using heatmaps. Basically, to plot such a two-variable function on a plane, the value of u at a point (x_0, y_0) is plotted as a color, instead of adding a third dimension in space. The red color spectrum represents higher values and the blue spectrum corresponds to smaller values.
For example the harmonic conjugate functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ can be plotted as follows for $\{z \in \mathbb{C} : -1 < x < 1, -1 < y < 1\}$ using the code given on the next page.
Modify the given code according to each function and plot each of the given function and its harmonic conjugate in MATLAB and submit all your graphs on Google Classroom.




```
[X Y] = meshgrid(-1:0.01:1);
U = X.^2-Y.^2;
V = 2*X.*Y;
subplot(121);
surf(X,Y,U,'EdgeColor','none');
view(2);
title('u(x,y) = x^2-y^2');
subplot(122);
surf(X,Y,V,'EdgeColor','none');
view(2);
title('u(x,y) = 2xy');
colormap jet;
```

Solution

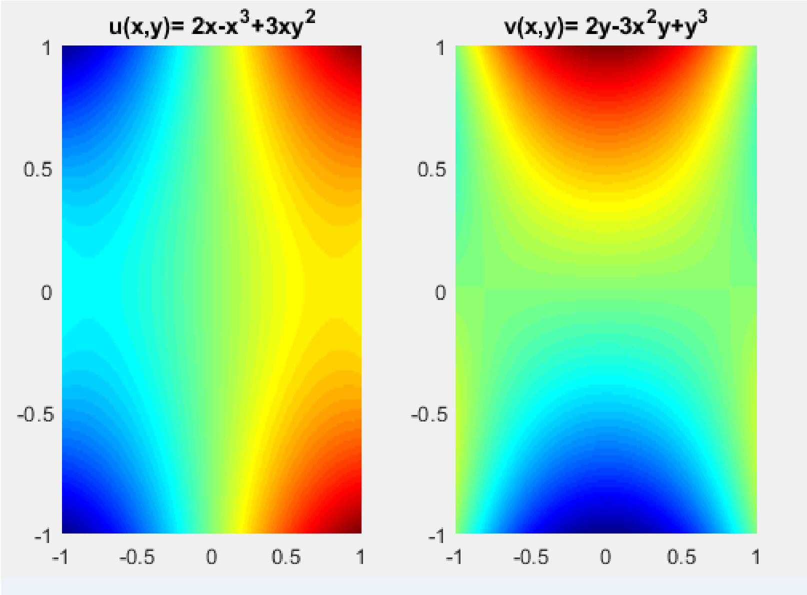
(a) $v(x,y) = 2y - 3x^2y + y^3$

(i) $v_x = -6xy \quad , \quad v_{xx} = -6y$
$v_y = 2 - 3x^2 + 3y^2 \quad , \quad v_{yy} = 6y$
$v_{xx} + v_{yy} = 0$
Function is Harmonic

(ii) $u_x = v_y$
$u_x = 2 - 3x^2 + 3y^2$
u = 2x - x^3 $+ 3xy^2 + g(y)$
$u_y = 6xy + g'(x) = -v_x$
$6xy + g'(x) = 6xy$
$g(x) = c$
$u(x,y) = 2x - x^3 + 3xy^2 + c$

(iii) $f(z) = u(x,y) + iv(x,y)$
$f(z) = (2x - x^3 + 3xy^2 + c) + i(2y - 3x^2y + y^3)$

(iv) Fig Below



(b) $u(x,y) = \dfrac{y}{x^2+y^2}$

(i) $u_x = \frac{-2xy}{(x^2+y^2)^2}$, $u_{xx} = \frac{(-2y)(-3x^2+y^2)}{(x^2+y^2)^2}$

$u_y = \frac{x^2-y^2}{(x^2+y^2)^2}$, $u_{yy} = \frac{(-2y)(3x^2-y^2)}{(x^2+y^2)^2}$

$u_{xx} + u_{yy} = 0$

Function is Harmonic

(ii) $u_x = v_y$

$v_y = \frac{-2xy}{(x^2+y^2)^2}$

$v = (\frac{x}{(x^2+y^2)^2}) + g(y)$

$v_x = (\frac{-x^2+y^2}{(x^2+y^2)^2}) + g'(x) = -u_x$

$g'(x) = 0$

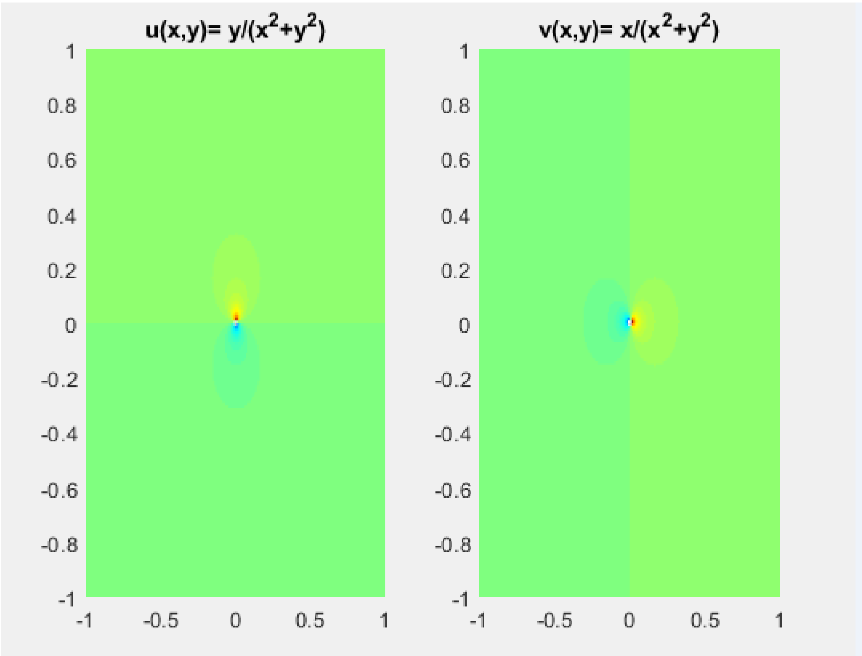
$g(x) = c$

$v(x,y) = \frac{x}{(x^2+y^2)} + c$

(iii) $f(z) = u(x,y) + iv(x,y)$

$f(z) = (\frac{y}{(x^2+y^2)}) + i(\frac{x}{(x^2+y^2)})$

(iv) Fig Below



(c) $u(x,y) = xy - x + y$

(i) $u_x = y - 1 \quad , \quad u_{xx} = 0$

$u_y = x + 1 \quad , \quad u_{yy} = 0$

$v_{xx} + v_{yy} = 0$

Function is Harmonic

(ii) $u_x = v_y$

$v_y = y - 1$

$v = \frac{y^2}{2} - y + g(y)$

$v_x = g'(x) = -u_y$

$g'(x) = -(x+1)$

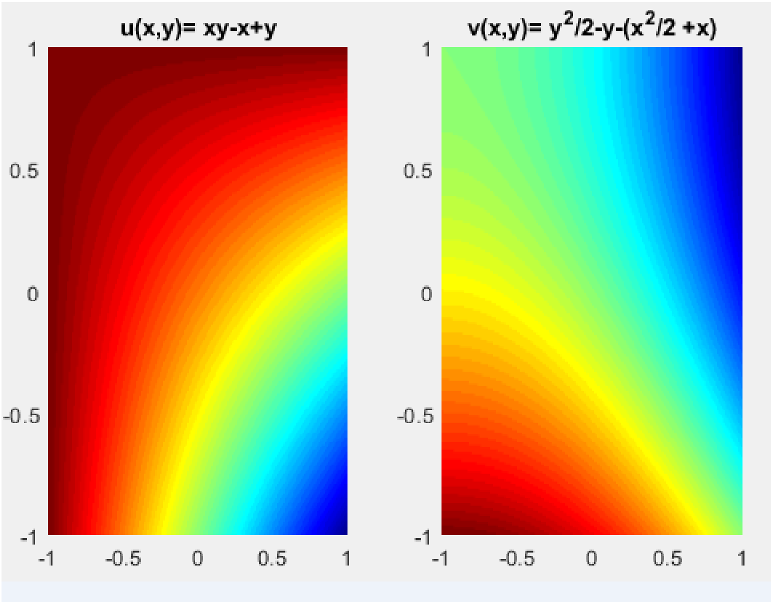
$g(x) = -(\frac{x^2}{2} + x) + c$

$v(x,y) = \frac{y^2}{2} - y - (\frac{x^2}{2} + x) + c$

(iii) $f(z) = u(x,y) + iv(x,y)$

$f(z) = (xy - x + y) + i(\frac{y^2}{2} - y - (\frac{x^2}{2} + x) + c)$

(iv) Fig Below



(d) $v(x,y) = x^2 - y^2 + 2y$

(i) $v_x = 2x \quad , \quad v_{xx} = 2$

$v_y = -2y + 2 \quad , \quad v_{yy} = -2$

$v_{xx} + v_{yy} = 0$

Function is Harmonic

(ii) $u_x = v_y$

$u_x = -2y + 2$

$u = -2xy + 2x + g(y)$

$u_y = -2x + g'(x) = -v_x$

$-2x + g'(x) = -2x$

$g'(x) = 0$

$g(x) = c$

$u(x,y) = -2xy + 2x$

(iii) $f(z) = u(x,y) + iv(x,y)$

$f(z) = (-2xy + 2x + c) + i(x^2 = y^2 + 2y)$

(iv) Fig Below