MT240: Complex Variables and Transforms

Homework 1 Solution

Due 2 pm, Thu Feb 22

Spring 2018

Tips to avoid a case of plagiarism:

- You must NOT look at the solutions of your classmates.
- Your are encouraged to discuss with your classmates, but restrict the discussions to using just your mouths and facial expressions.
- Cite all the online sources that you get help from.

Problem 1

Consider the complex number $z_0 = -4 - 4\sqrt{3}i$.

(a) (i) Plot it in the complex plane.

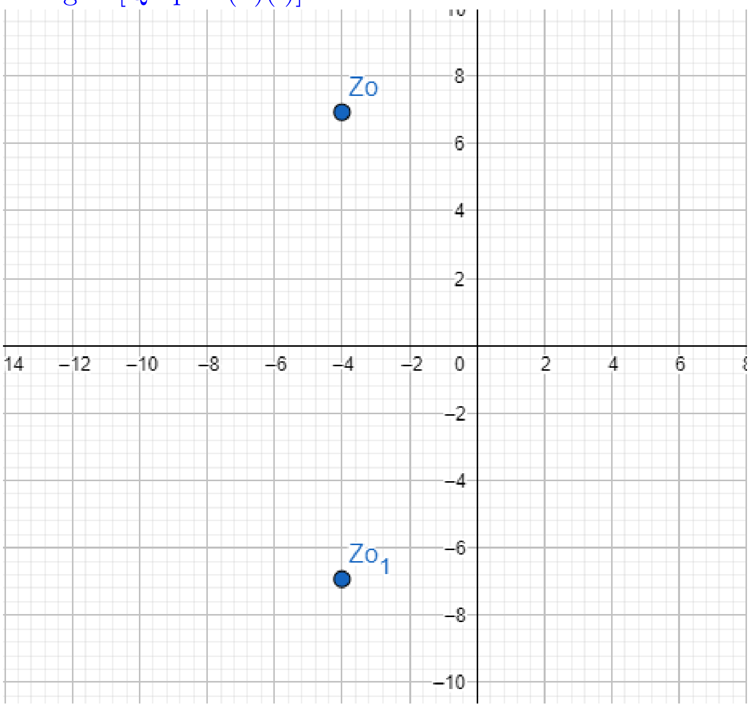
(ii) Find $|z_0|$ and Arg z_0, giving your answer both in degrees and radians.

(iii) Represent it in phasor, trigonometric and exponential forms.

(b) Repeat (a) for its complex conjugate $\overline{z_0}$.

Solution

(a) (i) subfigure[Q1 part(b)(i)]



(ii) $|z_0| = \sqrt{(-4)^2 + (-4\sqrt{3})^2} = 8$

Arg z_0=arctan$(\frac{-4\sqrt{3}}{-4}) = \dfrac{-2\pi}{3}$ $(or - 120°)$

(iii) Phasor: $8\angle - 120°$

Trignometric:$8(cos\frac{2\pi}{3} - isin\frac{2\pi}{3})$

Exponential: $8e^{-i\frac{2\pi}{3}}$

(b) (ii) $|z_0| = \sqrt{(-4)^2 + (4\sqrt{3})^2} = 8$

Arg z_0=arctan$(\frac{4\sqrt{3}}{-4}) = \frac{2\pi}{3}$ $(or 120°)$

(iiii) Phasor: $8\angle 120°$

Trignometric:$8(cos\frac{2\pi}{3} + isin\frac{2\pi}{3})$

Exponential: $8e^{i\frac{2\pi}{3}}$

(c) Given that z_0 is a root of a quadratic equation with real coefficients. Find that quadratic equation in the form $z^2 + bz + c = 0$.
(Hints: Conjugate root theorem. Factored form of quadratic equation or Vieta's formulas.)

Given that $z_0 = -4 - 4\sqrt{3}i$ is a root of the quadratic equation:$z^2 + bz + c = 0$ with real coefficients. Then using the conjugate root theorem we know that the second root of the equation will be $\overline{z_0} = -4 + 4\sqrt{3}i$. Using Vieta's Formula, for roots x_1, x_2:

$x_1 + x_2 = \frac{-b}{a}$
$z_0 + \overline{z_0} = \frac{-b}{a}$,
$(-4 + 4\sqrt{3}i) + (-4 - 4\sqrt{3}i) = \frac{-b}{a}$,
$\frac{-b}{a} = -8$
$\frac{b}{a} = 8$

$x_1 x_2 = \frac{c}{a}$
$z_0\overline{z_0} = \frac{c}{a}$
$(-4 + 4\sqrt{3}i)(-4 - 4\sqrt{3}i) = \frac{c}{a}$
$\frac{c}{a} = 64$

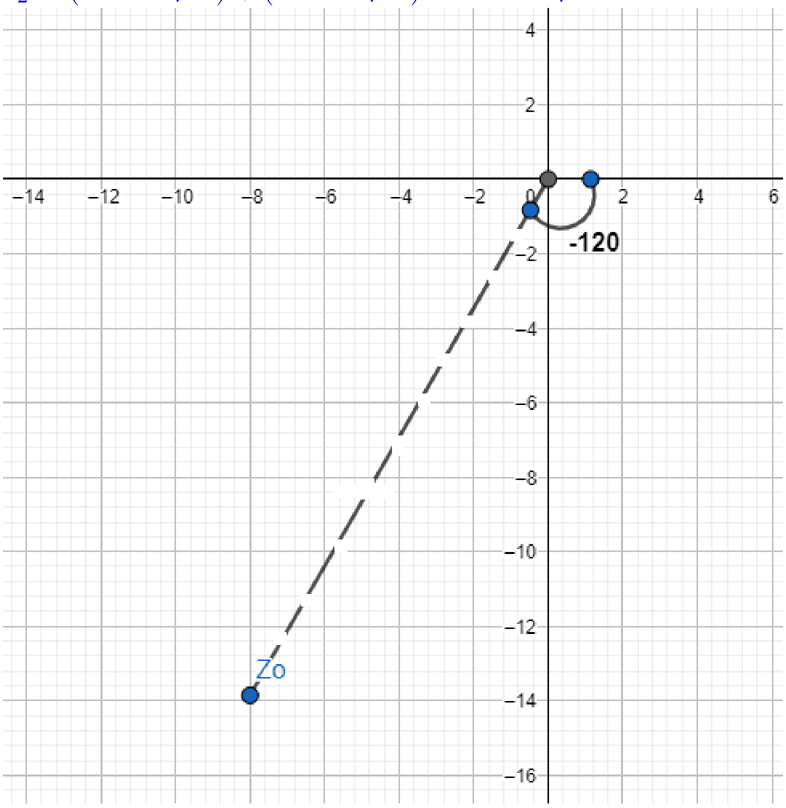
given $a = 1$
The Equation then becomes: $z^2 + 8z + 64 = 0$

(d) Now consider the complex number $z_1 = z_0 + z_0$.

(i) Evaluate z_1 and plot it in the complex plane.

(ii) Using your answers to part (a)(ii), find $|z_1|$ and Arg z_1.

(d) (i) $z_2 = (-4 - 4\sqrt{3}i) + (-4 - 4\sqrt{3}i) = -8 - 8\sqrt{3}i$



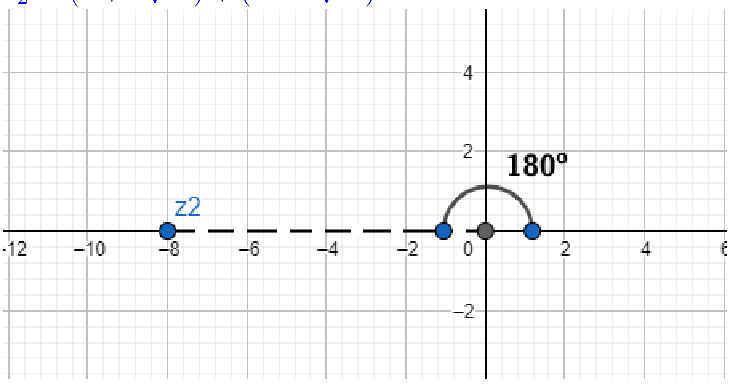
(ii) $z_1 = 2z_0 \ : \ |z_1| = 2|z_0| \ : \ |z_1| = 16$
$\text{Arg}z_1 = \text{Arg}z_0 = \ , \ \frac{-2\pi}{3} \ (or - 120°)$

(e) Now consider the complex number $z_2 = z_0 + \overline{z_0}$.

(i) Evaluate z_2 and plot it in the complex plane.

(ii) Using your plot, what is $|z_2|$ and Arg z_2?

(e) (i) $z_2 = (4 + 4\sqrt{3}i) + (4 - 4\sqrt{3}i) = -8$



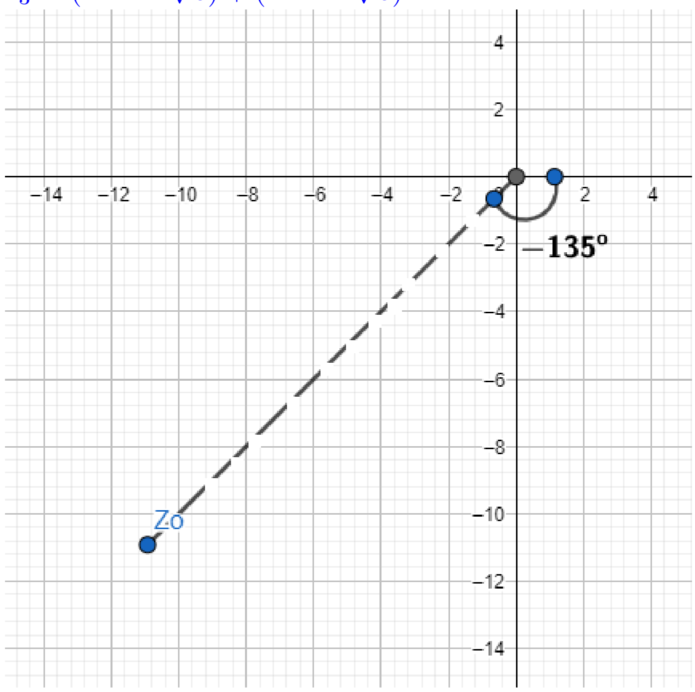
(ii) $|z_2| = Rez = 8 \ , \ \text{Arg}z_2 = 0°$

(f) Now consider the complex number $z_3 = z_0 + i\overline{z_0}$.

(i) Evaluate z_3 and plot it in the complex plane.

(ii) Using your plot, what is $|z_3|$ and Arg z_3?

(f) (i) $z_3 = (-4 - 4\sqrt{3}i) + (-4 + 4\sqrt{3}i) * i$
$z_3 = (-4 - 4\sqrt{3}) + (-4 - 4\sqrt{3})i$



(ii) $|z_3| = \sqrt{(-4 - 4\sqrt{3})^2 + -4 - 4\sqrt{3})^2} = 15.45$

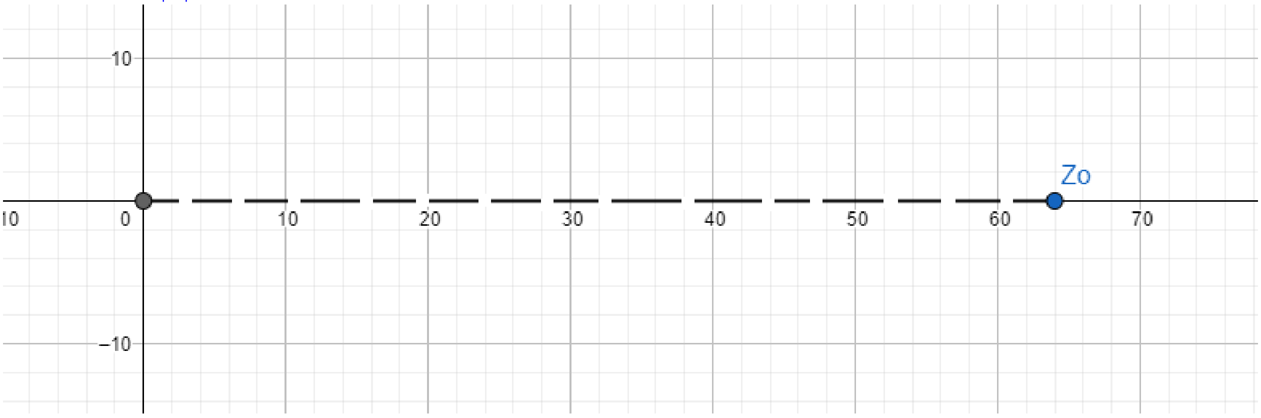
$\text{Arg } z_3\text{=arctan}\left(\frac{-4 - 4\sqrt{3}}{-4 - 4\sqrt{3}}\right) - \pi = \frac{-3\pi}{4} \ (or - 135°)$

(g) Now consider the complex number $z_4 = z_0\overline{z_0}$.

(i) Evaluate z_4 and plot it in the complex plane.

(ii) Using your answer to (a)(ii), find $|z_4|$ and Arg z_4.

(g) (i) $z_4 = z_0\overline{z_0} = |z|^2 = 8^2 = 64$



(ii) $|z_4| = 64$, $\text{Arg} z_4 = 0°$

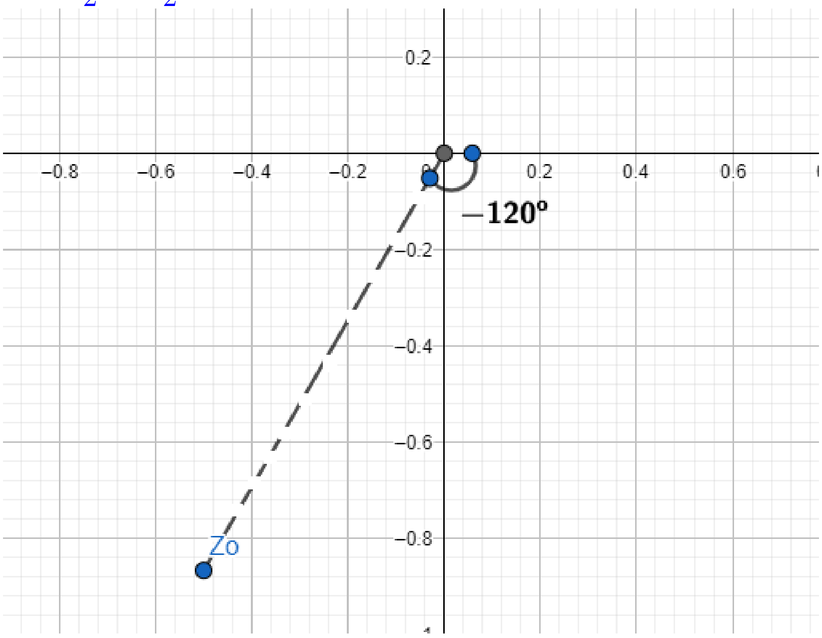
(h) Now consider the complex number $z_5 = \dfrac{\overline{z_0}}{z_0}$.

(i) Evaluate z_5 and plot it in the complex plane.

(ii) Using your answer to (a)(ii), find $|z_5|$ and Arg z_5.

(h) (i) $z_5 = \dfrac{-4+4\sqrt{3}}{--4\sqrt{3}} * \left(\dfrac{-4+4\sqrt{3}}{-4+4\sqrt{3}}\right)$

$z_5 = \dfrac{-1}{2} - \dfrac{\sqrt{3}}{2}i$

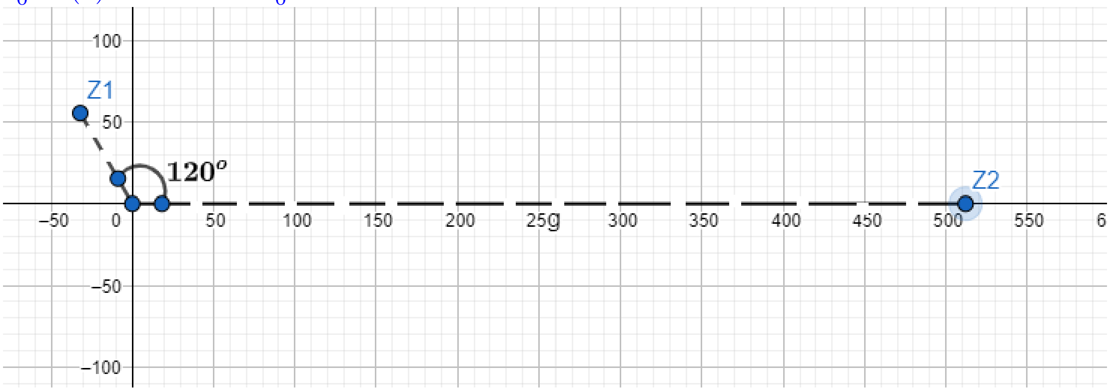


(ii) $|z_5| = \sqrt{\left(\dfrac{-1}{2}\right)^2 + \left(\dfrac{\sqrt{-3}}{2}\right)^2} = 1$

$\text{Arg } z_5 {=} \pi - \arctan\left(\dfrac{-0.5}{\dfrac{\sqrt{-3}}{2}}\right) = \dfrac{-2\pi}{3}\ (or - 120°)$

(i) Using your answer to (a)(iii), evaluate z_0^2 and z_0^3 and plot these in the complex plane.

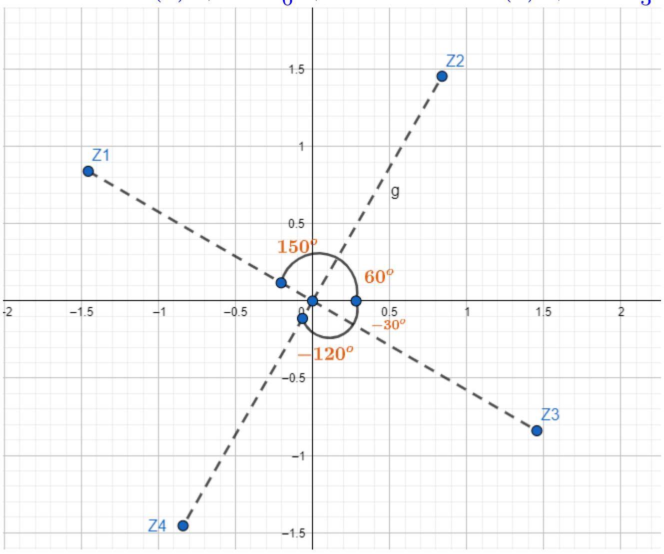
(i) $z_0 = 8e^{-i\frac{2\pi}{3}}$
$z_0^2 = (8)^2 e^{2\left(-i\frac{22\pi}{3}\right)} = z_0^2 = 64e^{-i\frac{4\pi}{3}}$
$z_0^3 = (8)^3 e^{3\left(-i\frac{2\pi}{3}\right)} = z_0^3 = 512e^{-i2\pi}$



(j) Solve the equation $z^4 - z_0 = 0$ and plot all the roots in the complex plane.

(j) $z^4 - z_0 = 0 \; ; \; z^4 = z_0 \; ; \; z = \left(8e^{-i\dfrac{2\pi}{3}}\right)^{\frac{1}{4}}$

$z = (8)^{\frac{1}{4}} e^{\left(-i\frac{\pi}{6} + \frac{n\pi}{2}\right)}$
$n = 0 : r = (8)^{\frac{1}{4}}, \theta = \frac{-\pi}{6}$, $n = 1 : r = (8)^{\frac{1}{4}}, \theta = \frac{\pi}{3}$
$n = 2 : r = (8)^{\frac{1}{4}}, \theta = \frac{5\pi}{6}$, $n = -1 : r = (8)^{\frac{1}{4}}, \theta = \frac{-2\pi}{3}$



(k) Find a complex number z_6 such that $|z_0 + z_6| = |z_0| + |z_6|$ where $z_6 \neq 0$ and $z_6 \neq z_0$.

The triangle inequality states that $|z_0 + z_6| = |z_0| + |z_6|$ holds true only when the vectors are parallel/on the same line. Hence using the rule that parallel vectors are multiples of one another we can write $z_6 = k * z_0$ where $k > 0$ and taking $k = 2$ we get $\boldsymbol{z_6 = -8 - 8\sqrt{3}i}$ which satisfies the equation.

Problem 2

Find all the complex solutions of the following algebraic polynomials and plot the solutions on the complex plane.

(a) $z^5 = 16\left(\sqrt{3} - i\right)$

(b) $z^8 - 1 = 0$

(c) $z^2 - 2z + i = 0$

(d) $z^3 - 3z^2 + 6z - 4 = 0$

Solution

(a) $z^5 = 16\left(\sqrt{3} - i\right)$

$z^5 = (32)e^{\left(-i\frac{\pi}{6} + \frac{n\pi}{2}\right)}$

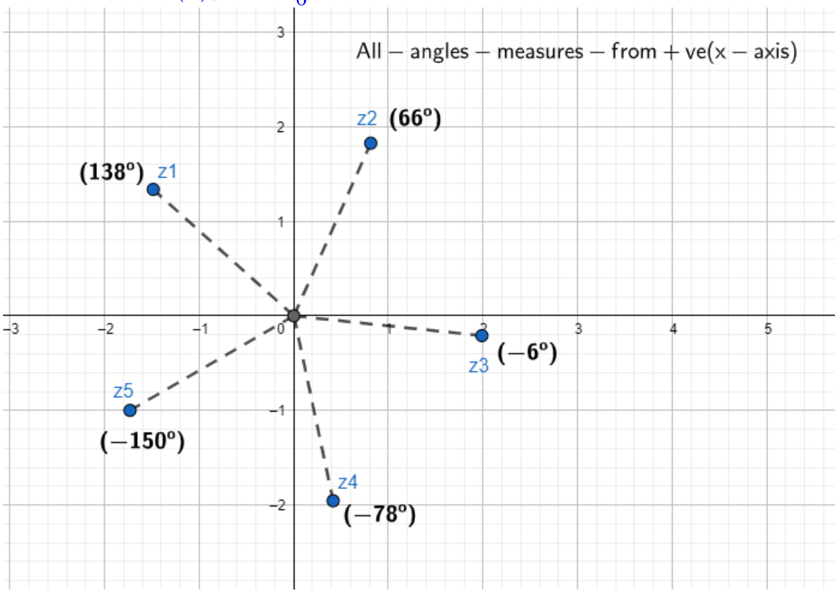
$z = (2)e^{\left(-i\frac{\pi}{6} + \frac{2n\pi}{5}\right)}$

n = -2 , -1 , 0 , 1 , 2

$n = 0 : r = (2), \theta = \frac{-\pi}{30}$, $n = 1 : r = (2), \theta = \frac{11\pi}{30}$

$n = 2 : r = (2), \theta = \frac{23\pi}{30}$, $n = -1 : r = (2), \theta = \frac{-13\pi}{30}$

$n = -2 : r = (2), \theta = \frac{-5\pi}{6}$



(b) $z^8 - 1 = 0$

$z^8 = 1$

$z^8 = (1)e^{(i\pi + 2n\pi)}$

$z = (1)e^{\left(-i\frac{\pi}{4}\right)}$

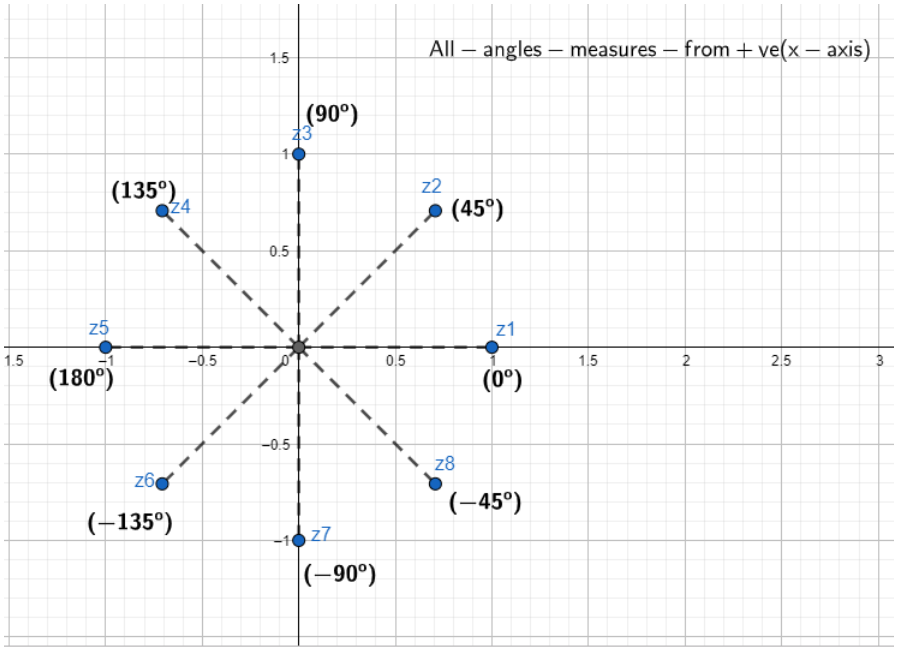
n = -3 , -2 , -1 , 0 , 1 , 2 , 3 , 4

$n = 0 : r = (1), \theta = 0$, $n = 1 : r = (1), \theta = \frac{\pi}{4}$

$n = 2 : r = (1), \theta = \frac{\pi}{2}$, $n = 3 : r = (1), \theta = \frac{3\pi}{4}$

$n = 4 : r = (1), \theta = \pi$, $n = -1 : r = (1), \theta = \frac{-\pi}{4}$

$n = -2 : r = (1), \theta = \frac{-\pi}{2}$, $n = -3 : r = (1), \theta = \frac{-3\pi}{4}$



(c) $z^2 - 2z + i = 0$

$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1,\ b = -2,\ c = i$

$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(i)}}{2(1)}$

$z = \frac{(2) \pm (2)\sqrt{(1-i)}}{2}$

$z_1 = 1 + \sqrt{1-i}$

$z_2 = 1 - \sqrt{1-i}$

(d) $z^3 - 3z^2 + 6z - 4 = 0$

$(z-1)(\frac{z^3 - 3z^2 + 6z - 4}{z-1}) = 0$

$(z-1)(z^2 - 2z + 4) = 0$

$z - 1 = 0$

$z_1 = 1$

$z^2 - 2z + 4 = 0$

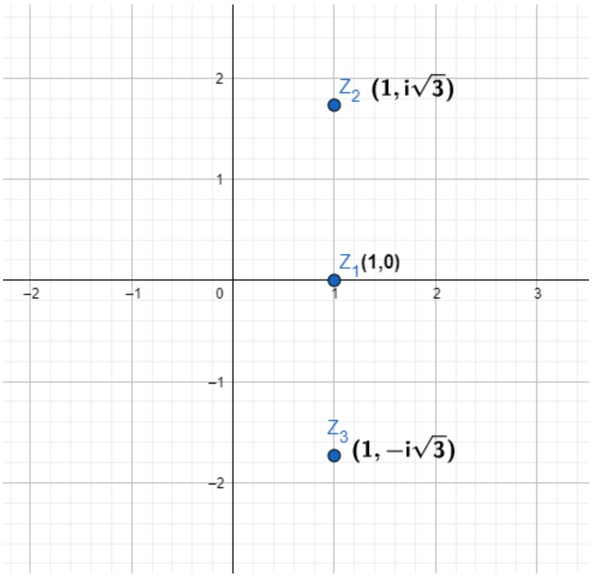
$a = 1,\ b = -2,\ c = 4$

$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$

$z = \frac{(2) \pm (2i)\sqrt{3}}{2}$

$z_2 = 1 + i\sqrt{3}$

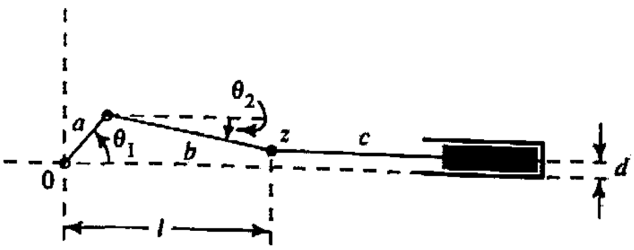
$z_3 = 1 - i\sqrt{3}$



Problem 3

This problem illustrates how complex notation can simplify the kinematic analysis of planar mechanisms.

Consider the crank-and-piston linkage depicted in the figure. The crank arm a rotates about the fixed point O while the piston arm c executes horizontal motion. (If this were a petrol engine, combustion forces would drive the piston and the connecting arm b would transform this energy into a rotation of the crankshaft.)



For engineering analysis, it is important to be able to relate the crankshaft's angular coordinates, i.e. position, velocity and acceleration, to the corresponding linear coordinates for the piston. Although this calculation can be carried out using vector analysis, the complex variable technique is more "automatic".

Let the crankshaft pivot O lie at the origin of the coordinate system, and let z be the complex number giving the location of the base of the piston rod, as depicted in the figure,

$$z = l + id$$

where l gives the piston's (linear) excursion and d is a fixed offset. The crank arm is described by $A = a(\cos\theta_1 + i\sin\theta_1)$ and the connecting arm by $B = b(\cos\theta_2 + i\sin\theta_2)$ (note that θ_2 is negative in the figure).

(a) Exploit the obvious identity $A+B = z = l+id$ to derive the following expression relating the piston position to the crankshaft angle:

$$l = a\cos\theta_1 + b\cos\left(\sin^{-1}\left(\frac{d - a\sin\theta_1}{b}\right)\right).$$

Solution

As we know

$z = l + id$

$A = a(\cos\theta_1 + i\sin\theta_1)$

$B = b(\cos\theta_2 + i\sin\theta_2)$

Also

$A + B = z = l + id$

Put A and B in above equation

$a(\cos\theta_1 + i\sin\theta_1) + b(\cos\theta_2 + i\sin\theta_2) = l + id$

$l =$a$(\cos\theta_1) + b(\cos\theta_2)$

$d =$a$(\sin\theta_1) + b(\sin\theta_2)$

$\theta_2 = (\sin^{-1}\left(\frac{d - a\sin\theta_1}{b}\right)$

Now we get

$$l = a\cos\theta_1 + b\cos\left(\sin^{-1}\left(\frac{d - a\sin\theta_1}{b}\right)\right).$$

(b) Suppose $a = 0.1, b = 0.2, d = 0.1$, and the crankshaft rotates at a uniform velocity of 2 rad/s. Compute the position and velocity of the piston when $\theta_1 = \pi$.

Position

$z = l + id$

$z = a\cos\theta_1 + b\cos\left(\sin^{-1}\left(\frac{d-a\sin\theta_1}{b}\right)\right) + id$

$z = 0.1\cos(\pi) + 0.2\cos\left(\sin^{-1}\left(\frac{0.1-0.1\sin(\pi)}{0.2}\right)\right) + i0.1$

$z = -0.1 + 0.2\cos\left(\sin^{-1}\left(\frac{0.1}{0.2}\right)\right) + i0.1$

$z = (\frac{-1+\sqrt{3}}{10}) + i(\frac{1}{10})$

Velocity

$l = a\cos\theta_1 + b\cos\left(\sin^{-1}\left(\frac{d-a\sin\theta_1}{b}\right)\right).$

$\frac{dl}{dt} = a(-\sin\theta_1)\frac{d\theta_1}{dt} + b - \sin\left(\sin^{-1}\left(\frac{d-a\sin\theta_1}{b}\right)\right)\frac{d}{dt}\sin^{-1}(\frac{d-a\sin\theta_1}{b})$

$\frac{dl}{dt} = a(-\sin\theta_1)\frac{d\theta_1}{dt} + b - \sin\left(\sin^{-1}\left(\frac{d-a\sin\theta_1}{b}\right)\right)\frac{1}{\sqrt{1-(\frac{d-a\sin\theta_1}{b})^2}}\frac{-a\cos\theta_1}{b}\frac{d\theta_1}{dt}$

$\frac{dl}{dt} = 0.1(-\sin\pi)(2) - 0.2\sin\left(\sin^{-1}\left(\frac{0.1-0.1\sin\pi}{0.2}\right)\right)\frac{1}{\sqrt{1-(\frac{0.1-0.1\sin\pi}{0.2})^2}}\frac{-0.1\cos\pi}{0.2}(2)$

$\frac{dl}{dt} = -0.1156$

Problem 4

Show that for all $z \in \mathbb{C}$, $\overline{e^z} = e^{\overline{z}}$.

Solution

$\overline{e^z} = \overline{e^x cisy}$
$e^x(\cos y - i \sin y)$
$e^x(\cos -y - i \sin -y)$
$e^x e^{-iy}$
$e^{\overline{z}}$

Problem 5

For each of the following sets, answer the following with the help of suitable definitions.

(a) $\{z \in \mathbb{C} : 0.1 < |z - 2| < 3\}$

(b) $\{z \in \mathbb{C} : 0 < |z - 2| < 3\}$

(c) $\{z \in \mathbb{C} : 0 < \text{Re } z \leq 1\} \cap \{z \in \mathbb{C} : 1 < \text{Im } z < 2\}$

(d) $\{z \in \mathbb{C} : 0 < \text{Re } z \leq 1\} \cup \{z \in \mathbb{C} : 1 < \text{Im } z < 2\}$

(e) $\left\{z \in \mathbb{C} : \text{Re}\left(z^2\right) > 1\right\}$

(f) $\left\{z \in \mathbb{C} : -\frac{\pi}{4} < \text{Arg } z < \frac{\pi}{4}\right\} \setminus \text{K}_1(3)$

For each of the above sets, answer the following.

(i) Sketch the set in the complex plane.

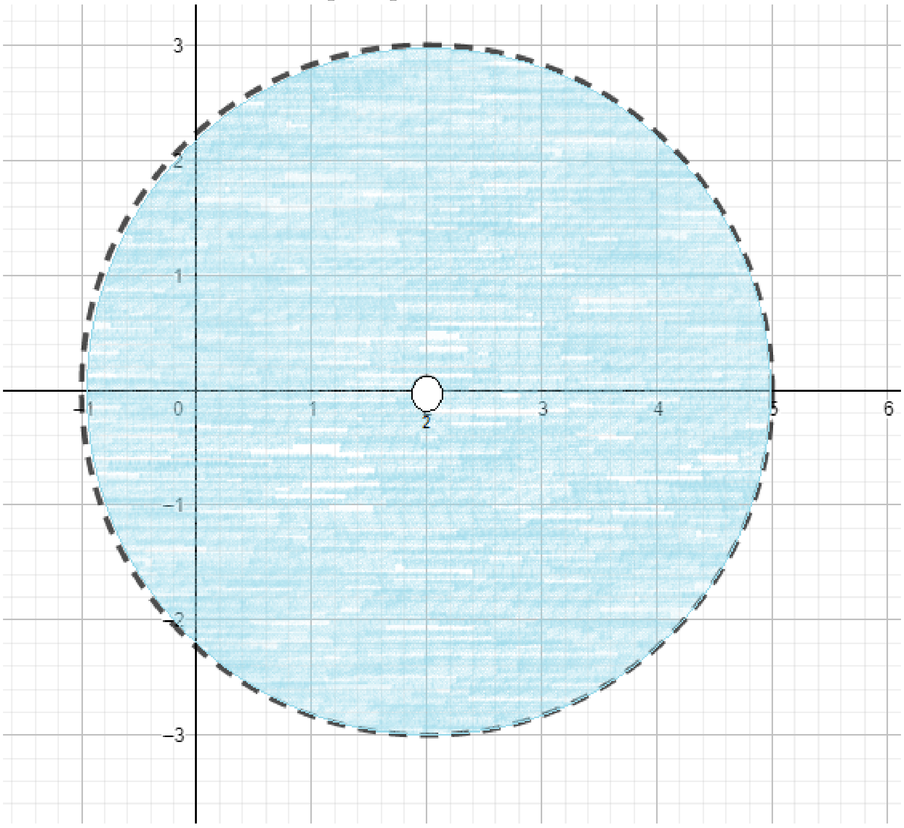
(ii) Is the set bounded?

(iii) Is the set open, closed or neither?

(iv) Write down the closure of the set in set-builder notation.

(v) If the set is open, is it connected?

a (i) Sketch the set in the complex plane.



(ii) Is the set bounded?

It is bounded because there exists R such that, for $R > 0$; E $\subset \mathrm{B}_R(0)$

(iii) Is the set open, closed or neither?

The set is open because is contains all of it's interior points.

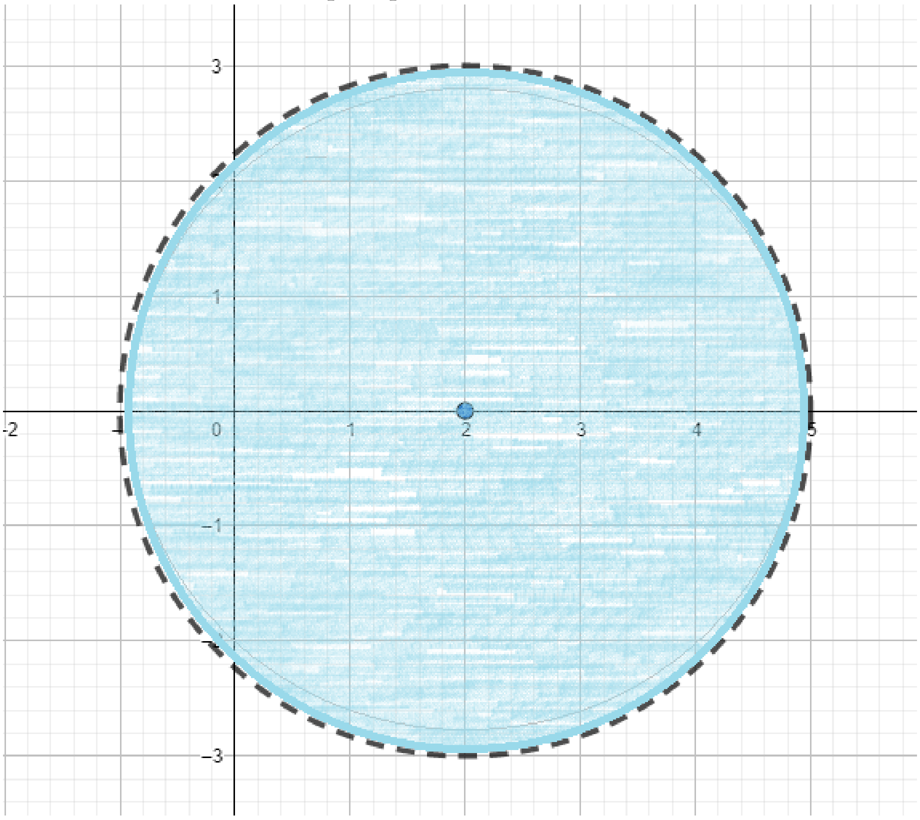
(iv) Write down the closure of the set in set-builder notation.

$\{z \in \mathbb{C} : 0.1 < |z - 2| < 3\}$

(v) If the set is open, is it connected?

Yes

b (i) Sketch the set in the complex plane.



(ii) Is the set bounded?

It is bounded because there exists R such that, for $R > 0$; E $\subset$B$_R(0)$

(iii) Is the set open, closed or neither?

Closed

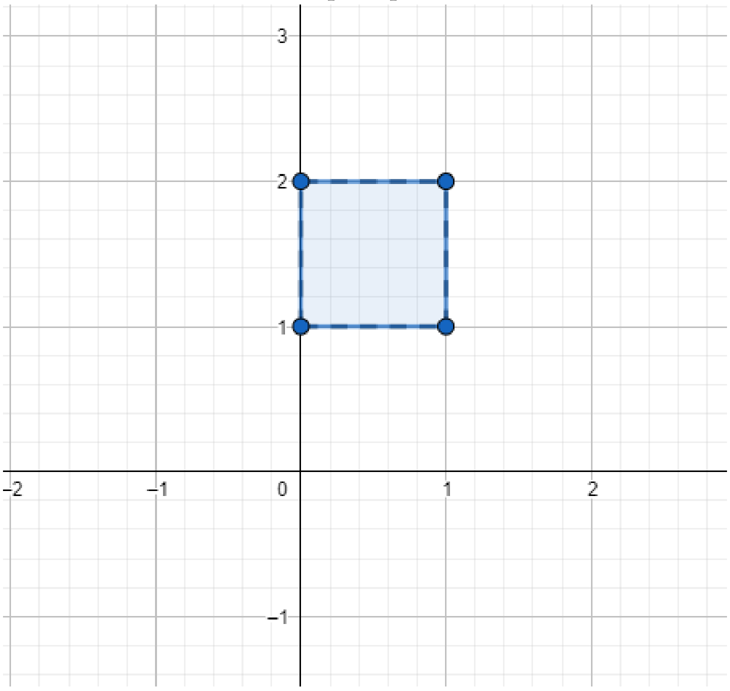
(iv) Write down the closure of the set in set-builder notation.

$\{z \in \mathbb{C} : 0 < \text{Re } z \leq 1\} \cap \{z \in \mathbb{C} : 1 < \text{Im } z < 2\}$

(v) If the set is open, is it connected?

NO

c (i) Sketch the set in the complex plane.



(ii) Is the set bounded?

It is bounded because there exists R such that, for $R > 0$; E $\subset B_R(0)$

(iii) Is the set open, closed or neither?

The set is open because is contains all of it's interior points.

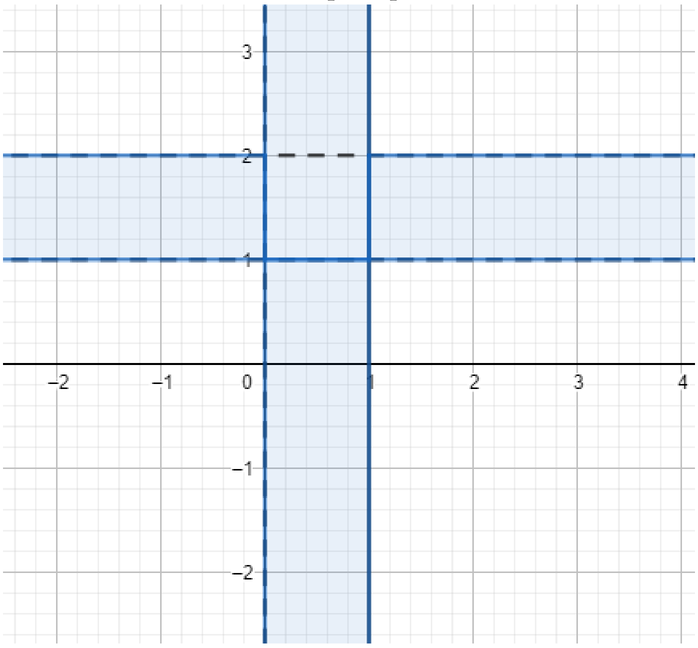
(iv) Write down the closure of the set in set-builder notation.

$\{z \in \mathbb{C} : 0 < \text{Re } z \leq 1\} \cap \{z \in \mathbb{C} : 1 < \text{Im } z < 2\}$

(v) If the set is open, is it connected?

Yes

d (i) Sketch the set in the complex plane.



(ii) Is the set bounded?

It is unbounded because there exists no R such that, for $R > 0$; E $\subset B_R(0)$

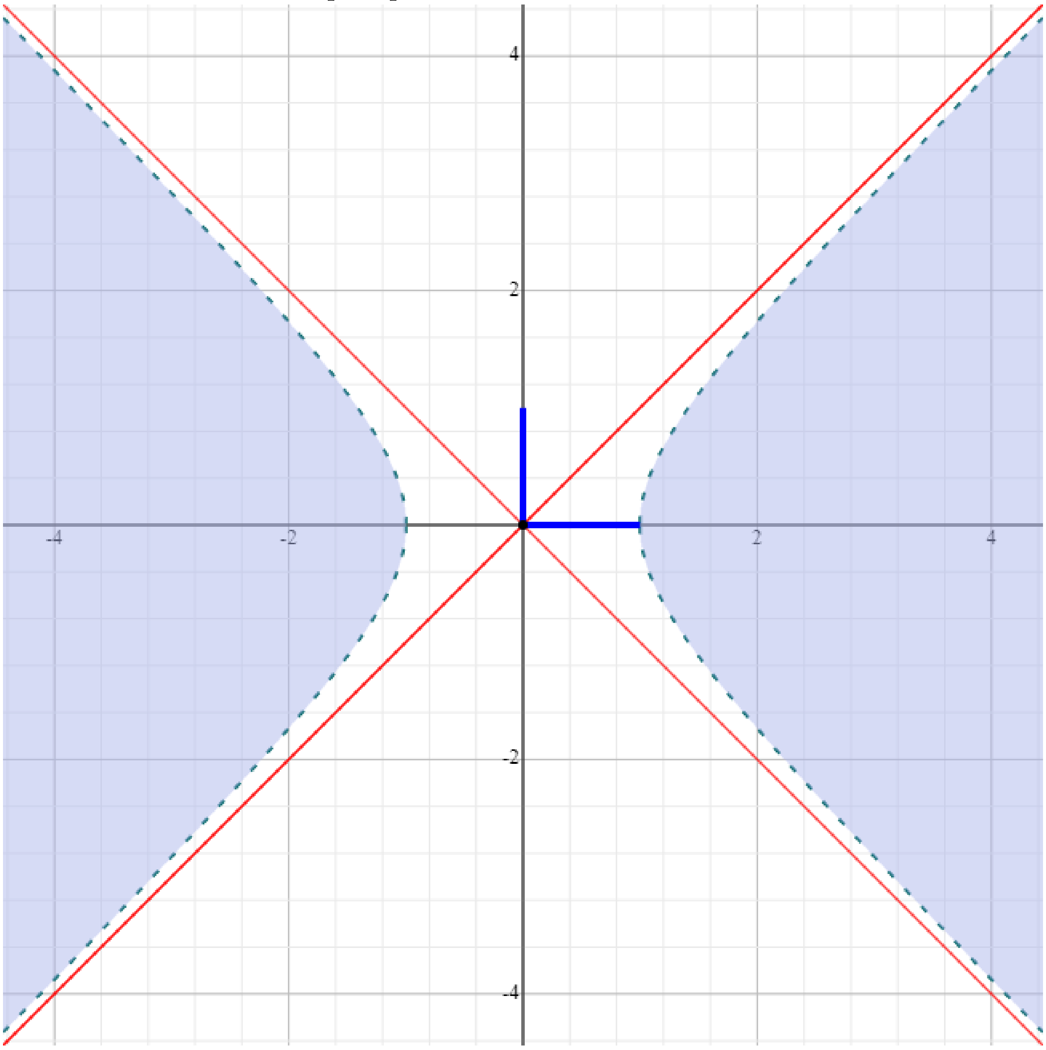
(iii) Is the set open, closed or neither?

The set is open because is contains all of it's interior points.

(iv) Write down the closure of the set in set-builder notation.
$\{z \in \mathbb{C} : 0 < \text{Re } z \leq 1\} \cup \{z \in \mathbb{C} : 1 < \text{Im } z < 2\}$

(v) If the set is open, is it connected?
Yes

e (i) Sketch the set in the complex plane.



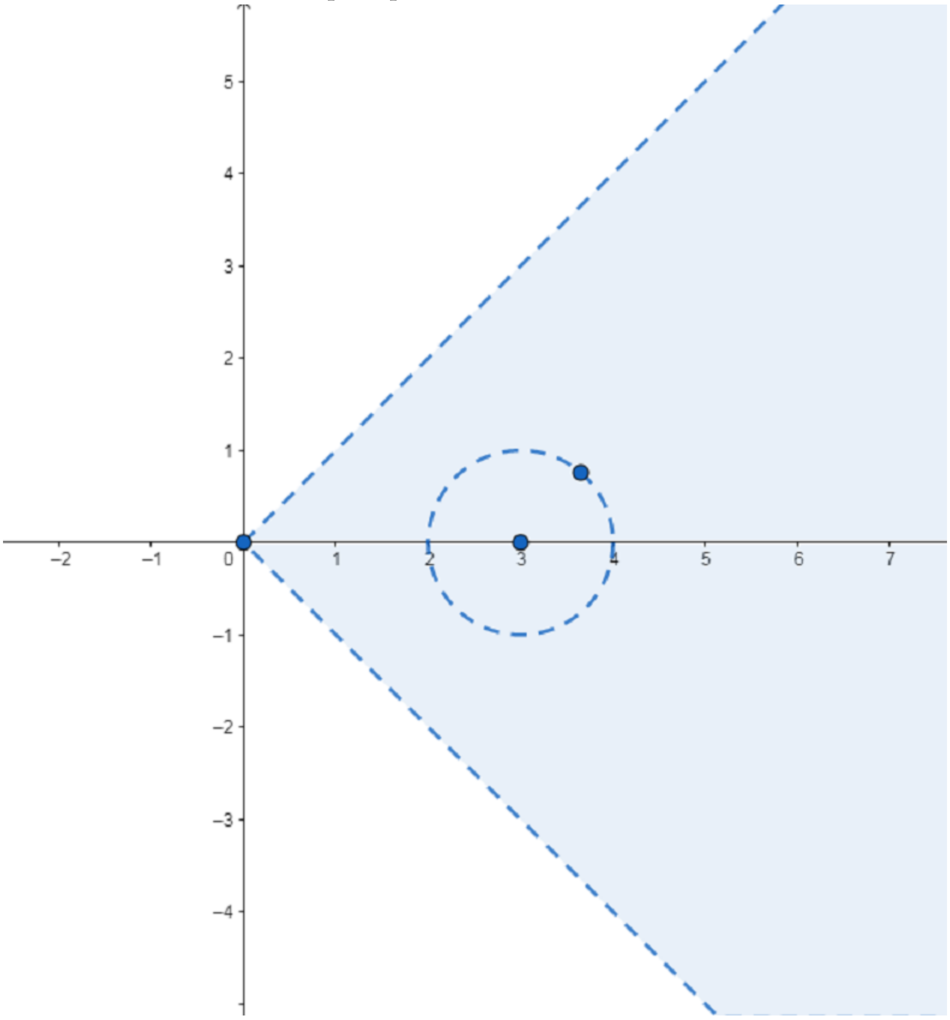
(ii) Is the set bounded?
It is bounded because there exists R such that, for $R > 0$; E $\subset$B$_R(0)$

(iii) Is the set open, closed or neither?
Neither

(iv) Write down the closure of the set in set-builder notation.
$\left\{z \in \mathbb{C} : \text{Re}\left(z^2\right) > 1\right\}$

(v) If the set is open, is it connected?
NO

f (i) Sketch the set in the complex plane.



(ii) Is the set bounded?

It is unbounded because there exists no R such that, for $R > 0$; E $\subset$B$_R(0)$

(iii) Is the set open, closed or neither?

The set is open because is contains all of it's interior points.

The is not close because it does not contain all it's boundary points

(iv) Write down the closure of the set in set-builder notation.

$\left\{z \in \mathbb{C} : -\frac{\pi}{4} < \text{Arg}\, z < \frac{\pi}{4}\right\} \setminus \mathrm{K}_1(3)$

(v) If the set is open, is it connected?

The set is open hence but not connected. Because a point from interior of circle $K_1(3)$ cannot be connected by a line segment to a point in the set outside the circle