MT240: Complex Variables and Transforms

Homework 2

Due 8 am, Fri Mar 8

Spring 2019

Tips to avoid plagiarism

- Do not copy the solutions of your classmates.
- Your are encouraged to discuss the problems with your classmates in whatever way you like but make sure to REPRODUCE YOUR OWN SOLUTIONS in what you submit for grading.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1

For each of the following functions, sketch the image of the given set and represent the image in the set-builder notation.

Tip: First map the boundary of domain to the u-v plane. Then choose an interior point of the domain, map it to u-v plane and see on what side of the boundary does it get mapped in the u-v plane.

(a) $f(z) = z + 1 + 2i$, for $\{z : |z| < 1\}$

(b) $f(z) = 2z$, for $\{z : |z| < 1\}$

(c) $f(z) = \left(1 - i\sqrt{3}\right) z$, for $\{z : |z| < 1 \quad \wedge \quad \text{Im } z > 0\}$ [Hint: Convert $1 - i\sqrt{3}$ to the polar form]

(d) $f(z) = \left(1 - i\sqrt{3}\right) z + 1 + 2i$, for $\{z : |z| < 1 \quad \wedge \quad \text{Im } z > 0\}$

(e) $f(z) = \dfrac{1}{z}$, for $\left\{z : \frac{1}{2} < |z| < 1\right\}$ [Hint: Polar form]

(f) $f(z) = z^3$, for $\{z : \text{Re } z = -\text{Im } z\}$

(g) $f(z) = e^{iz}$ for $\{z : -1 < \text{Re } z < 1 \quad \wedge \quad 0 < \text{Im } z < 2\}$

Problem 2

Evaluate the following limits and write your answer in the form $x + iy$. If the limit exists, determine whether the function is continuous or not at this point.

In case the limit does not exist, state the reason.

(a) $\lim\limits_{z \to 4i} \dfrac{z}{z^2 + 16}$ (b) $\lim\limits_{z \to i} \dfrac{z^2 + 1}{z^4 - 1}$ (c) $\lim\limits_{z \to 2i} \dfrac{z^2 + 4}{z}$ (d) $\lim\limits_{z \to -1} \text{Arg } z$

Problem 3

Find a complex function that maps the lower half-plane to the 2nd quadrant.

[Hint: First find a complex function that maps the lower half-plane to the 4th quadrant. Then think about how to map the 4th quadrant to the 2nd quadrant.]

Problem 4

For each of the following functions

(a) $f(z) = xy + iy$

(b) $f(z) = iz^2$

(c) $f(z) = 2xy + i(x^2 - y^2)$

(d) $f(z) = e^{iz}$

(e) $f(z) = e^x e^{iy}$

(f) $f(z) = \ln r + i\theta$

(g) $f(z) = z + \dfrac{1}{z}$ [Hint: Polar]

(h) $f(z) = x^3 + i(1 - y)^3$

(i) Find the set in the complex plane on which the function is analytic.

(ii) In case a function is analytic on some set, find $f'(z)$ in that set.

(iii) Determine the singular points of the function.

Problem 5

For each of the following functions,

(a) $v(x, y) = 2y - 3x^2y + y^3$

(b) $u(x, y) = \dfrac{y}{x^2 + y^2}$ [Hint: Polar]

(c) $u(x, y) = xy - x + y$

(d) $v(x, y) = x^2 - y^2 + 2y$

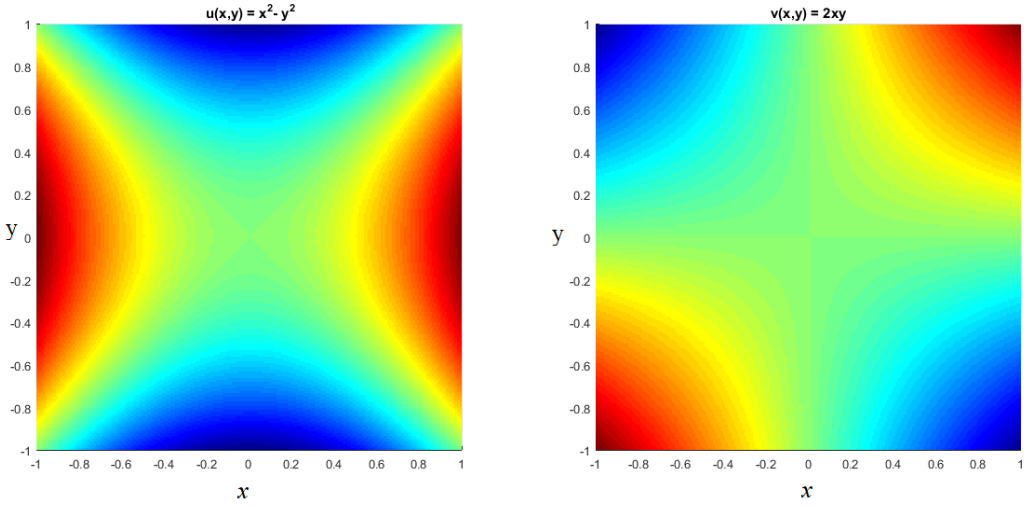
(i) Determine whether the function is harmonic.

(ii) If it is harmonic, find its harmonic conjugate function.

(iii) Form an analytic function $f(z) = u(x, y) + iv(x, y)$ from the harmonic function and its harmonic conjugate.

(iv) Each of the functions $u(x, y)$ and $v(x, y)$ can be plotted in MATLAB using heatmaps. Basically, to plot such a two-variable function on a plane, the value of u at a point (x_0, y_0) is plotted as a color, instead of adding a third dimension in space. The red color spectrum represents higher values and the blue spectrum corresponds to smaller values.

For example the harmonic conjugate functions $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ can be plotted as follows for $\{z \in \mathbb{C} : -1 < x < 1, -1 < y < 1\}$ using the code given on the next page.



Modify the given code according to each function and plot each of the given function and its harmonic conjugate in MATLAB and submit all your graphs on Google Classroom.

```
[X Y] = meshgrid(-1:0.01:1);
U = X.^2-Y.^2;
V = 2*X.*Y;
subplot(121);
surf(X,Y,U,'EdgeColor','none');
view(2);
title('u(x,y) = x^2-y^2');
subplot(122);
surf(X,Y,V,'EdgeColor','none');
view(2);
title('u(x,y) = 2xy');
colormap jet;
```