MT240: Complex Variables and Transforms

Homework 1

Due 8 am, Fri Feb 22 *Spring 2019*

Tips to avoid plagiarism

- Do not copy the solutions of your classmates.
- Your are encouraged to discuss the problems with your classmates in whatever way you like but make sure to REPRODUCE YOUR OWN SOLUTIONS in what you submit for grading.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1

Consider the complex number $z_0 = -4 - 4\sqrt{3}i$.

(a) (i) Plot it in the complex plane.
  (ii) Find $|z_0|$ and Arg z_0, giving your answer both in degrees and radians.
  (iii) Represent it in phasor, trigonometric and exponential forms.

(b) Repeat (a) for its complex conjugate $\overline{z_0}$.

(c) Given that z_0 is a root of a quadratic equation with real coefficients. Find that quadratic equation in the form $z^2 + bz + c = 0$.
(Hints: Conjugate root theorem. Factored form of quadratic equation or Vieta's formulas.)

(d) Now consider the complex number $z_1 = z_0 + z_0$.
  (i) Evaluate z_1 and plot it in the complex plane.
  (ii) Using your answers to part (a)(ii), find $|z_1|$ and Arg z_1.

(e) Now consider the complex number $z_2 = z_0 + \overline{z_0}$.
  (i) Evaluate z_2 and plot it in the complex plane.
  (ii) Using your plot, what is $|z_2|$ and Arg z_2?

(f) Now consider the complex number $z_3 = z_0 + i\overline{z_0}$.
  (i) Evaluate z_3 and plot it in the complex plane.
  (ii) Using your plot, what is $|z_3|$ and Arg z_3?

(g) Now consider the complex number $z_4 = z_0\overline{z_0}$.
  (i) Evaluate z_4 and plot it in the complex plane.
  (ii) Using your answer to (a)(ii), find $|z_4|$ and Arg z_4.

(h) Now consider the complex number $z_5 = \dfrac{\overline{z_0}}{z_0}$.
  (i) Evaluate z_5 and plot it in the complex plane.
  (ii) Using your answer to (a)(ii), find $|z_5|$ and Arg z_5.

(i) Using your answer to (a)(iii), evaluate z_0^2 and z_0^3 and plot these in the complex plane.

(j) Solve the equation $z^4 - z_0 = 0$ and plot all the roots in the complex plane.

(k) Find a complex number z_6 such that $|z_0 + z_6| = |z_0| + |z_6|$ where $z_6 \neq 0$ and $z_6 \neq z_0$.

Problem 2

Find all the complex solutions of the following algebraic polynomials and plot the solutions on the complex plane.

(a) $z^5 = 16\left(\sqrt{3} - i\right)$

(b) $z^8 - 1 = 0$

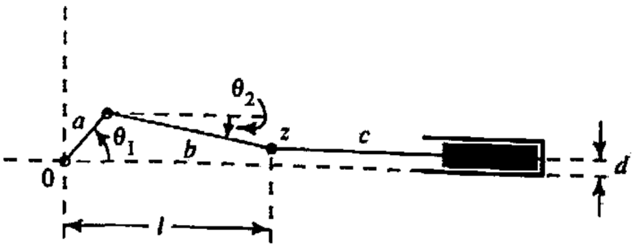
(c) $z^2 - 2z + i = 0$

(d) $z^3 - 3z^2 + 6z - 4 = 0$

Problem 3

This problem illustrates how complex notation can simplify the kinematic analysis of planar mechanisms.

Consider the crank-and-piston linkage depicted in the figure. The crank arm a rotates about the fixed point O while the piston arm c executes horizontal motion. (If this were a petrol engine, combustion forces would drive the piston and the connecting arm b would transform this energy into a rotation of the crankshaft.)



For engineering analysis, it is important to be able to relate the crankshaft's angular coordinates, i.e. position, velocity and acceleration, to the corresponding linear coordinates for the piston. Although this calculation can be carried out using vector analysis, the complex variable technique is more "automatic".

Let the crankshaft pivot O lie at the origin of the coordinate system, and let z be the complex number giving the location of the base of the piston rod, as depicted in the figure,

$$z = l + id$$

where l gives the piston's (linear) excursion and d is a fixed offset. The crank arm is described by $A = a(\cos\theta_1 + i\sin\theta_1)$ and the connecting arm by $B = b(\cos\theta_2 + i\sin\theta_2)$ (note that θ_2 is negative in the figure).

(a) Exploit the obvious identity $A + B = z = l + id$ to derive the following expression relating the piston position to the crankshaft angle:

$$l = a\cos\theta_1 + b\cos\left(\sin^{-1}\left(\frac{d - a\sin\theta_1}{b}\right)\right).$$

(b) Suppose $a = 0.1, b = 0.2, d = 0.1$, and the crankshaft rotates at a uniform velocity of 2 rad/s. Compute the position and velocity of the piston when $\theta_1 = \pi$.

Problem 4

Show that for all $z \in \mathbb{C}$, $\overline{e^z} = e^{\overline{z}}$.

Problem 5

For each of the following sets, answer the following with the help of suitable definitions.

(a) $\{z \in \mathbb{C} : 0.1 < |z-2| < 3\}$

(b) $\{z \in \mathbb{C} : 0 < |z-2| < 3\}$

(c) $\{z \in \mathbb{C} : 0 < \operatorname{Re} z \leq 1\} \cap \{z \in \mathbb{C} : 1 < \operatorname{Im} z < 2\}$

(d) $\{z \in \mathbb{C} : 0 < \operatorname{Re} z \leq 1\} \cup \{z \in \mathbb{C} : 1 < \operatorname{Im} z < 2\}$

(e) $\left\{z \in \mathbb{C} : \operatorname{Re}\left(z^2\right) > 1\right\}$

(f) $\left\{z \in \mathbb{C} : -\frac{\pi}{4} < \operatorname{Arg} z < \frac{\pi}{4}\right\} \setminus \mathrm{K}_1(3)$

For each of the above sets, answer the following.

(i) Sketch the set in the complex plane.

(ii) Is the set bounded?

(iii) Is the set open, closed or neither?

(iv) Write down the closure of the set in set-builder notation.

(v) If the set is open, is it connected?