MT240: Complex Variables and Transforms

Final Exam (Spring 2019)

Wednesday, May 22

Name: ________________________________ **Roll Number:** ________________

180 Minutes

Instructions

- There are **20** printed pages and **6** blank page in this booklet.
- **All problems** are compulsory.
- **Calculators** are strictly **not allowed**.
- Write **all your work in this booklet**, including any rough work.
- Read the statement **carefully** before you start attempting a problem.
- Properly **label all the axes and relevant points** if you draw any graphs.
- You are allowed to get help from **your own hard copy of lecture notes** uploaded on Google Classroom.
- This exam will assess your following **Course Learning Objectives** (CLOs)
  - CLO 1: Determine whether a complex function is analytic.
  - CLO 2: Calculate the mapping through a complex analytic function.
  - CLO 3: Evaluate the integrals related to Fourier and Laplace transforms for standard functions and interpret their graphs.

Problem	**1**	**2**	**3**	**4**	**5**	**Total**
Marks	15	30	25	20	10	**100**
	CLO 1	CLO 3				

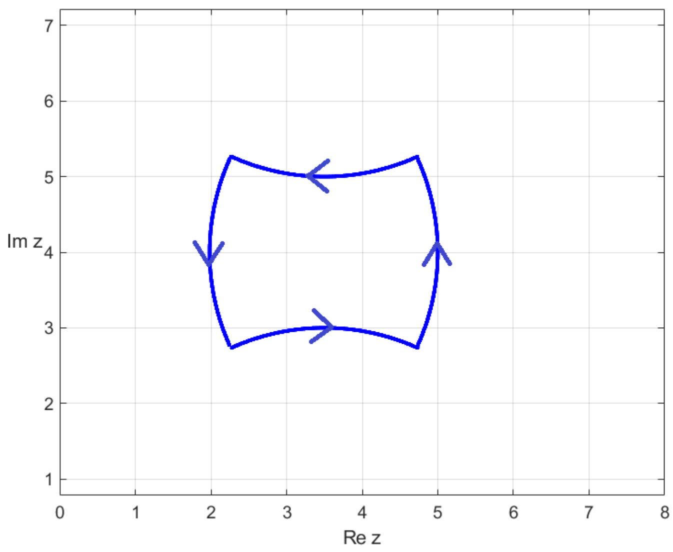
Course Instructor: Usama Bin Sikandar

Page for marks and contestation.
Do NOT write anything on this page.

P1	**P2**	**P3**	**P4**	**P5**	**Total**
15	**30**	**25**	**20**	**10**	**100**

Problem 1 [15 marks]

Consider a closed contour γ shown in the figure below.



Evaluate the following integrals.

(a) $\oint_\gamma \frac{z-3-4i}{z}\, dz$

$\oint_\gamma \frac{z-3-4i}{z}\, dz = 0,$ **Using Cauchy's Integral Theorem**

(b) $\oint_\gamma \frac{z}{z-3-4i}\, dz$

$= 2\pi i(3+4i) = -8\pi + 6\pi i$

(c) $\oint_\gamma \frac{z^2}{(z-3-4i)^2}\ dz$

$= \frac{2\pi i}{1!}.\frac{d}{dz}z^2|_{3+4i} = 2\pi i.(2z)|_{3+4i} = -16\pi + 12\pi i$

(d) $\oint_\gamma \frac{z}{(z-4-4i)(z-3-4i)^2}\ dz$

$Res(f, 4+4i) = \frac{z}{(z-3-4i)^2}|_{4+4i} = 4+4i$

$Res(f, 3+4i) = \frac{d}{dz}\frac{z}{z-4-4i}|_{3+4i} = \frac{(z-4-4i)-z}{(z-4-4i)^2}|_{3+4i} = -4-4i$

$\oint_\gamma \frac{z^2}{(z-3-4i)^2}\ dz = 2\pi i(Res(f, 4+4i) + (Res(f, 3+4i))) = 0$

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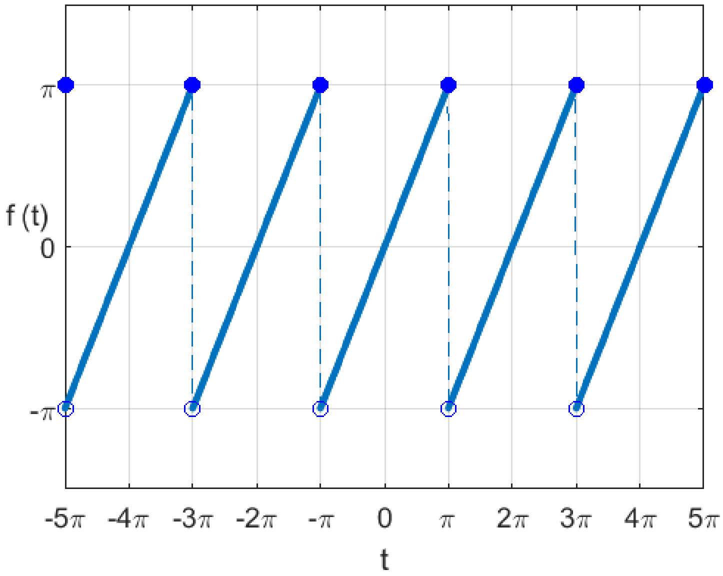
Problem 2 [30 marks]

(a) Consider the periodic function $f(t)$ shown in the figure.



(i) What is the time period of $f(t)$?
2π

(ii) Is the function even or odd? Explain.
ODD
$f(t) = -f(-t)$

(iii) Based on (ii), which of the following is true about the complex Fourier series coefficients of $f(t)$?
$c_n = -c_{-n}$

(iv) Write down an expression for $f(t)$ in terms of t for $-\frac{T}{2} < t \leq \frac{T}{2}$.
$f(t) = t, \qquad t[-\pi, \pi]$

(v) Evaluate its complex Fourier series coefficients c_n.

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-inw_ot}\ dt$

$= \frac{1}{2\pi}\int_{0}^{\pi} t(-2isinnt)\ dt$

$= \left.\frac{itcos(nt)}{n\pi}\right|_0^\pi - \left.\frac{itsin(nt)}{n^2\pi})\right|_0^\pi$

$= \frac{i(-1)^n}{n}, \qquad n \neq 0$

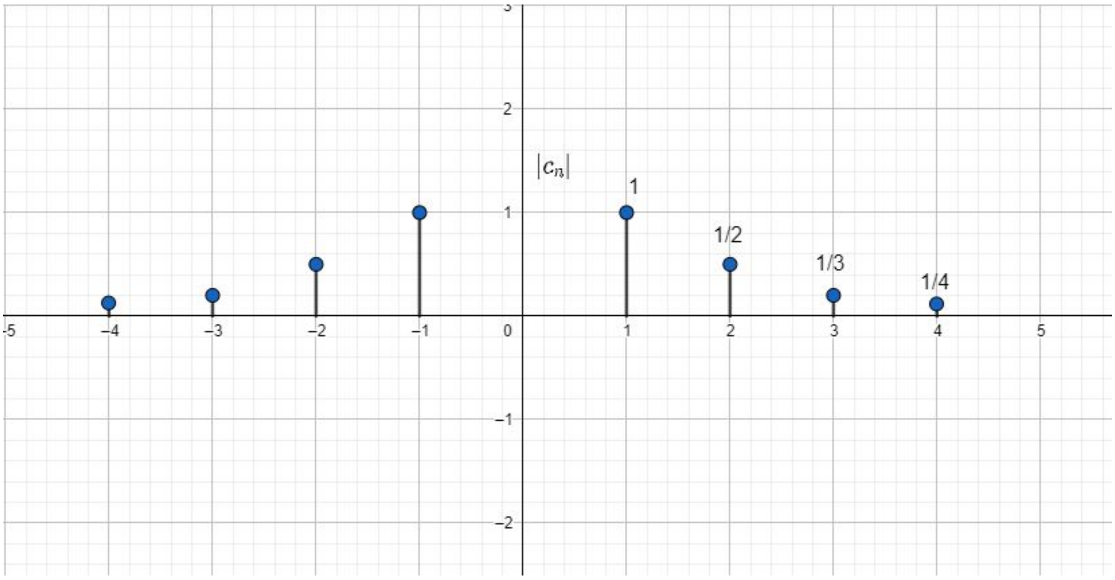
$c_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} (t)\ dt = 0$

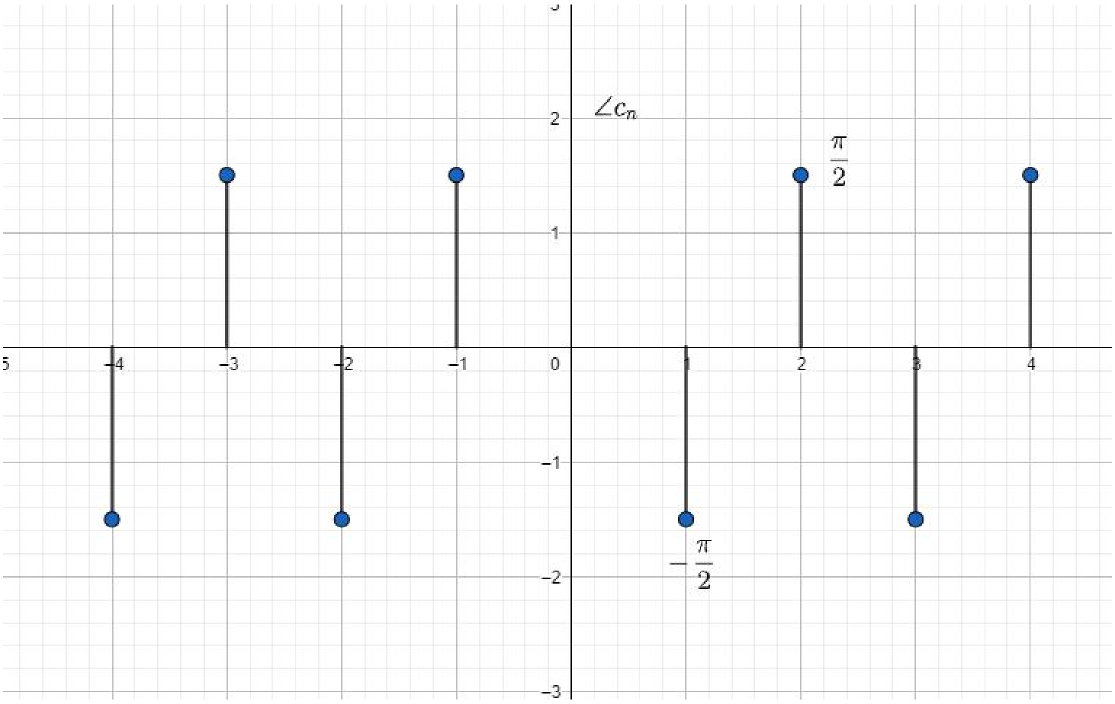
(vi) Using your answer to (v), find a_n and b_n, the coefficients of Fourier cosine series and sine series respectively.

$a_n = c_n + \overline{c_n} = 0$

$b_n = i(c_n + \overline{c_n}) = -\dfrac{2(-1)^n}{n}$

(vii) Plot the amplitude spectrum $|c_n|$ and phase spectrum $\angle c_n$ on the axes given below for $n = [-4 \quad -3 \quad -2 \quad -1 \;\; 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4]$.





(viii) Use Parseval's identity to evaluate the summation $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

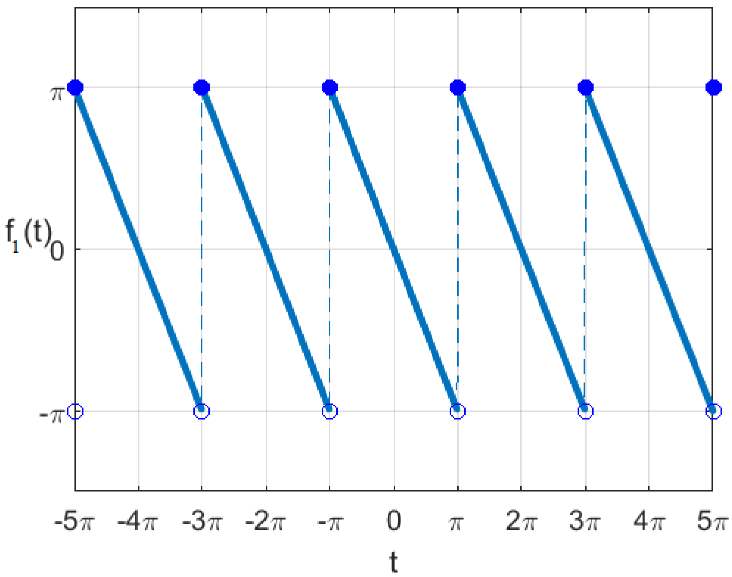
$\frac{1}{2\pi} \int_{-\pi}^{\pi} |t|^2 \, dt = \sum_{n=-\infty, n\neq 0}^{\infty} |\frac{i(-1)^n}{n}|^2$

$\frac{t^3}{3\pi}|_0^\pi = \sum_{n=1}^{\infty} \frac{2}{n^2}$

$\frac{\pi^2}{3} = \sum_{n=1}^{\infty} \frac{2}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(b) Using your answer to a(v), find the complex Fourier series coefficients d_n of the following function $f_1(t)$.



(a) $f_1(t) = -f(t)$

$d_n = -c_n$

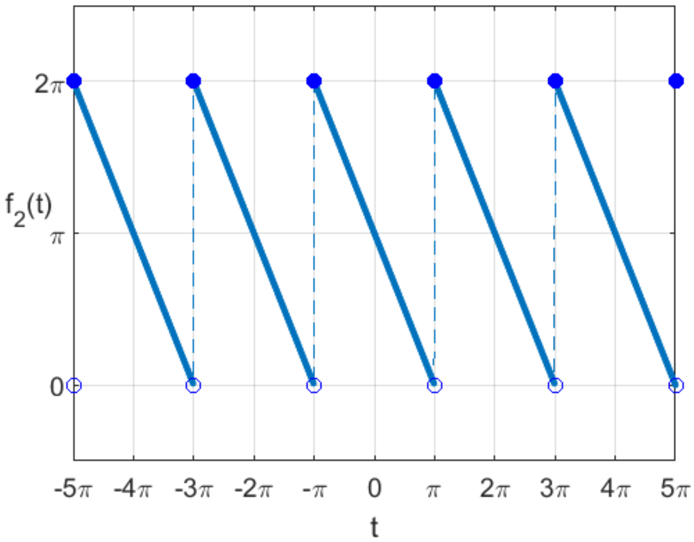
(b) $f_1(t) = f(-t)$

$d_n = c_n$

$$d_n = \frac{-i(-1)^n}{n} \qquad n \neq 0$$

$d_o = 0$

(c) (i) Using your answer to a(v), find the complex Fourier series coefficients e_n of the following function $f_2(t)$.



$f_2(t) = -f(t) + \pi$

$e_n = \frac{-i(-1)^n}{n}$ n$\neq 0$

$e_o = \frac{1}{2\pi} \int_{-\pi}^{\pi} (-f(t) + \pi)$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \pi = \pi$

(ii) Find the value of Fourier series of $f_2(t)$ at $t = -\frac{3}{2}T$.

$t = -3\pi$

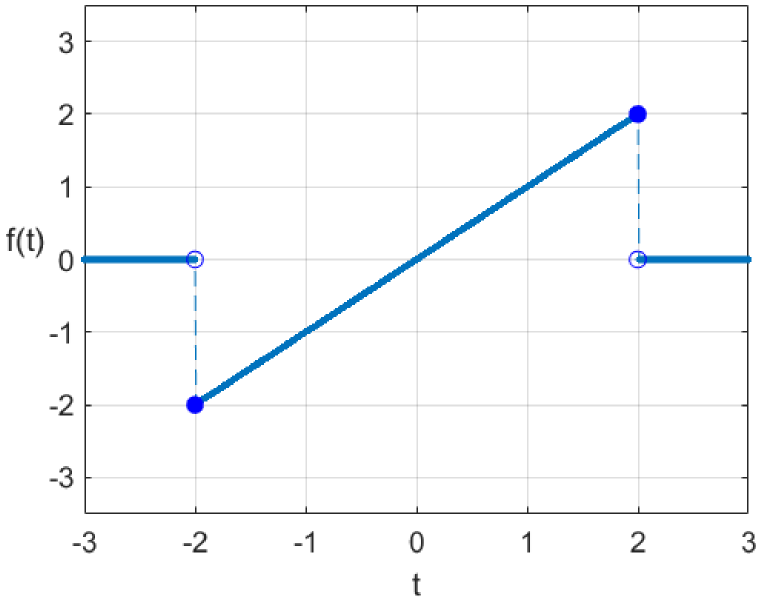
$f_2(t) = \frac{f_2(-3\pi^-) + f_2(-3\pi^+)}{2}$

$f_2(t) = \frac{0 + 2\pi}{2}$

$f_2(t) = \pi$

Problem 3 [25 marks]

Consider the function $f(t)$ shown in the figure below.



(a) Is this function even or odd? Explain.
ODD

$f(t) = -f(-t)$

(b) Based on (a), is its Fourier transform $F(\omega)$ real or pure imaginary?
Pure Imaginary

(c) Evaluate $F(\omega)$.

$F(w) = \int_{-2}^{2} te^{-iwt}\ dt$

$F(w) = -2i \int_{0}^{2} tsinwt\ dt$

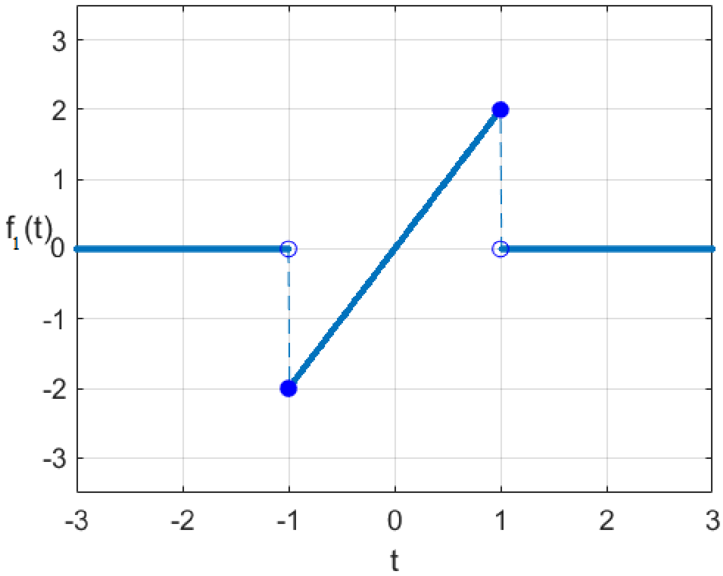
$$F(w) = \left.\frac{2itcos(wt)}{w}\right|_0^2 - \left.\frac{2itsin(wt)}{w^2}\right|_0^2$$

$$F(w) = \frac{i4tcos(2w)}{w} - \frac{2isin(2w)}{w^2}$$

$$F(w) = \frac{i8tcos(2w)}{2w} - \frac{i8sin(2w)}{2w^2} \quad \text{w} \neq 0$$

$F(0) = \int_{-2}^{2} t\ dt = 0$

(d) Using your answer to (c), find the Fourier transform of the following function $f_1(t)$.

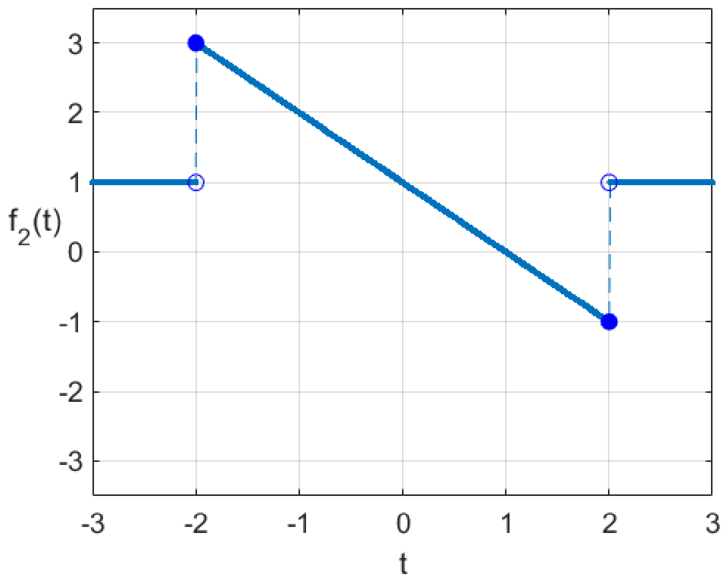


$f_1(t) = f(2t)$

$F_1(w) = \frac{1}{2}F(\frac{w}{2})$

$F_1(w) = \frac{4i}{w}\left(cos(w) - \frac{sin(w)}{w}\right)$

(e) Using your answer to (c), find the Fourier transform of the following function $f_2(t)$.

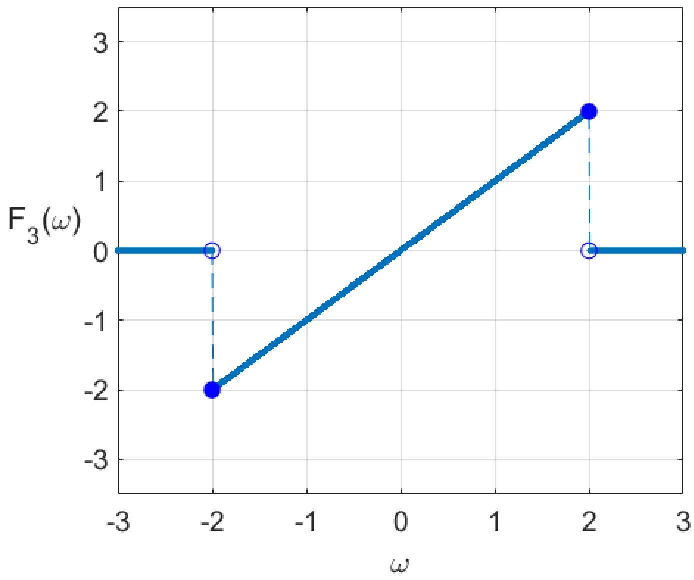


$f_2(t) = f(-t) + 1$

$F_2(w) = F(-w) + 2\pi\delta(w)$

$F_2(w) = i\left[\frac{2sin(2w)}{w^2} - \frac{4cos(2w)}{w}\right] + 2\pi\delta(w)$

(f) The Fourier transform of a function $f_3(t)$ is shown in the figure below. Using your answer to (c), find the function $f_3(t)$.



$F_3(w) = f(w)$

$f(t) \longleftrightarrow F(w)$

$F(t) \longleftrightarrow 2(-w)$

$\frac{1}{2\pi}F(t) \longleftrightarrow f(-w)$

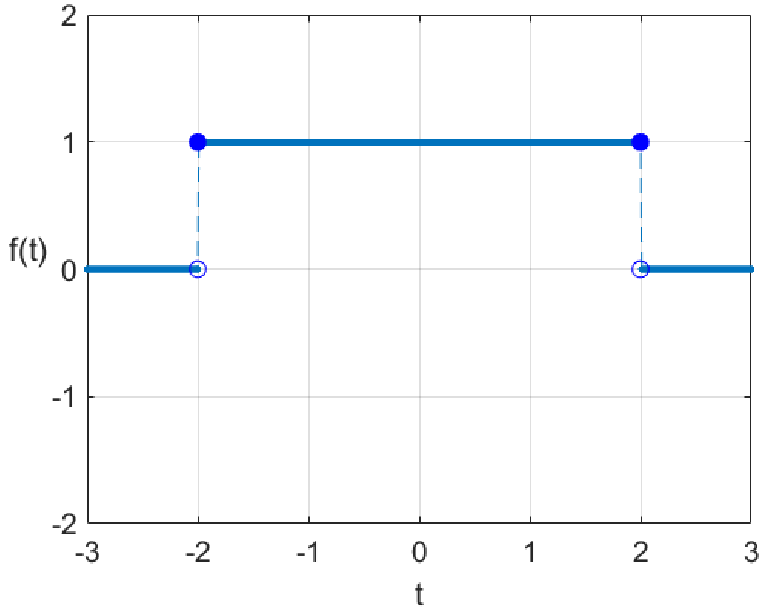
$\frac{1}{2\pi}F(-t) \longleftrightarrow F_3(w)$

$$f_3(t) = i\left[\frac{sin(t)}{\pi t^2} - \frac{2cos(2t)}{\pi t}\right]$$

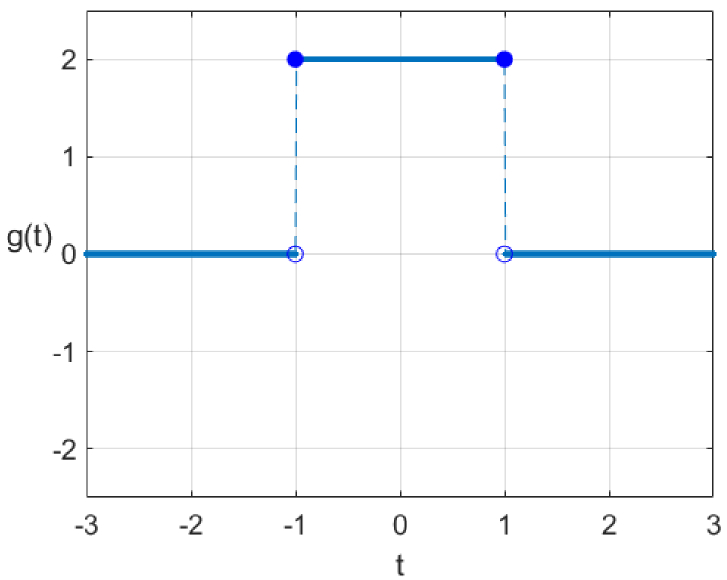
(g) Using the shape of $F_3(\omega)$, determine whether $f_3(t)$ is absoltely integrable.
No, because $F_3(w)$ is not continuous

Problem 4 [20 marks]

Consider the function $f(t)$ shown in the figure.



(a) Evaluate the convolution product $f(t) * g(t)$, where $g(t)$ is shown below.



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Solution

for: $|t| \geq -3$

$$f(t) * g(t) = 0$$

for: $-3 < t < -1$

$$\int_{-2}^{t+1} 2 \ dt$$
$$= 2t + 6$$

for: $-1 < t < 1$

$$\int_{t-1}^{t+1} 2 \ dt$$
$$= 4$$

for: $1 < t < 3$

$$\int_{t-1}^{2} 2 \ dt$$
$$= 6 - 2t$$

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \ dt = \begin{cases} 0, & |t| \geq 3 \\ 6 + 2t, & -3 < t \leq -1 \\ 4, & -1 \leq t \leq 1 \\ 6 - 2t, & 1 < t < 3 \end{cases}$$

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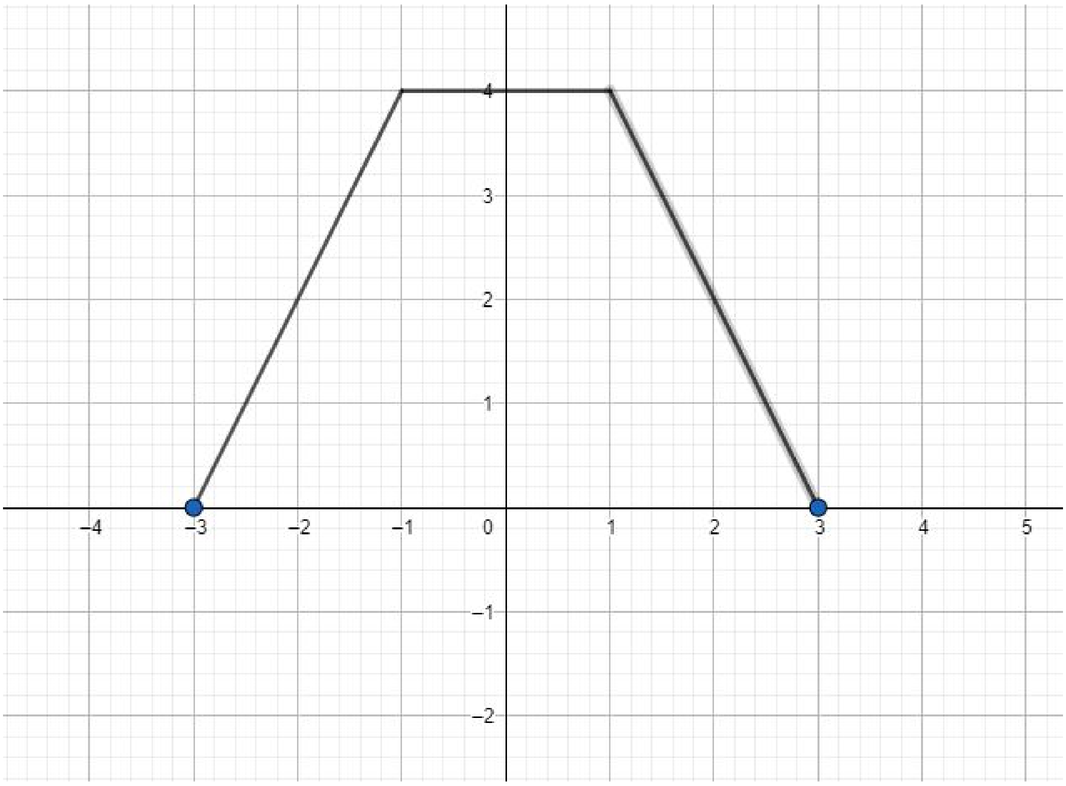
(b) Sketch the graph of $f(t) * g(t)$.



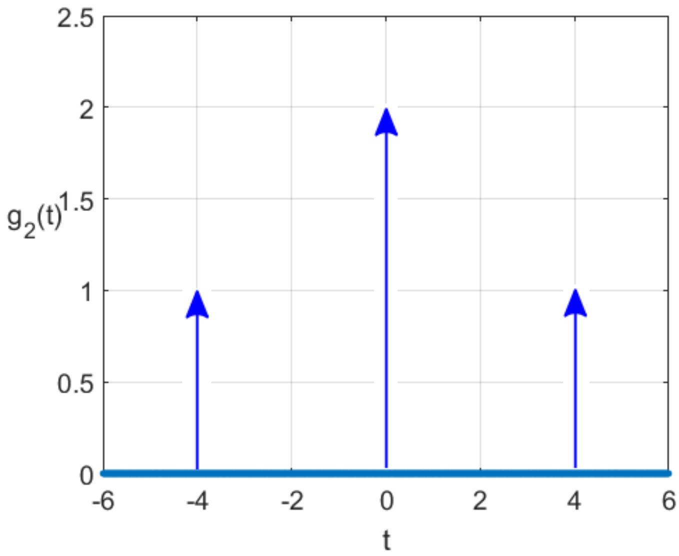
(c) Without evaluating any integral, find the Fourier transform of $f(t) * g(t)$.

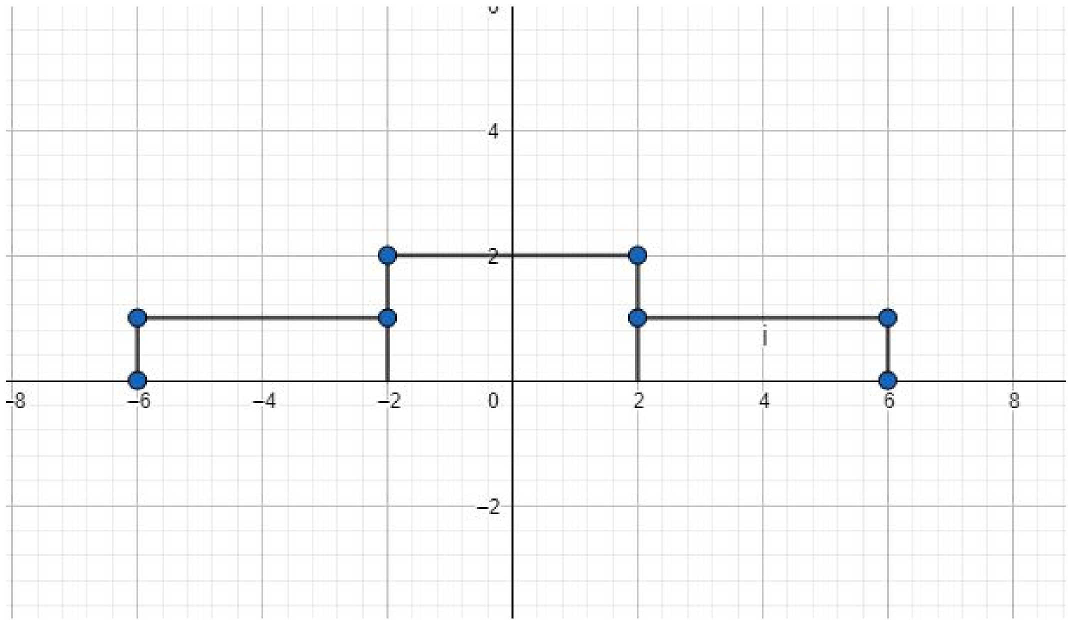
$$f(t) * g(t) \longleftrightarrow F(w)G(w)$$

$$F(w)G(w) = \left[\frac{4sin(\frac{4w}{2})}{\frac{4w}{2}}\right]\left[\frac{4sin(\frac{2w}{2})}{\frac{2w}{2}}\right]$$

$$F(w)G(w) = \frac{8sin(2w)sin(w)}{w^2}$$

(d) Sketch the graph of convolution product $f(t) * g_2(t)$, where $g_2(t)$ is shown below.





(e) Without evaluating any integral, find the Fourier transform of $f(t) * g_2(t)$.

$f(t) * g(t) \longleftrightarrow F(w)G(w)$

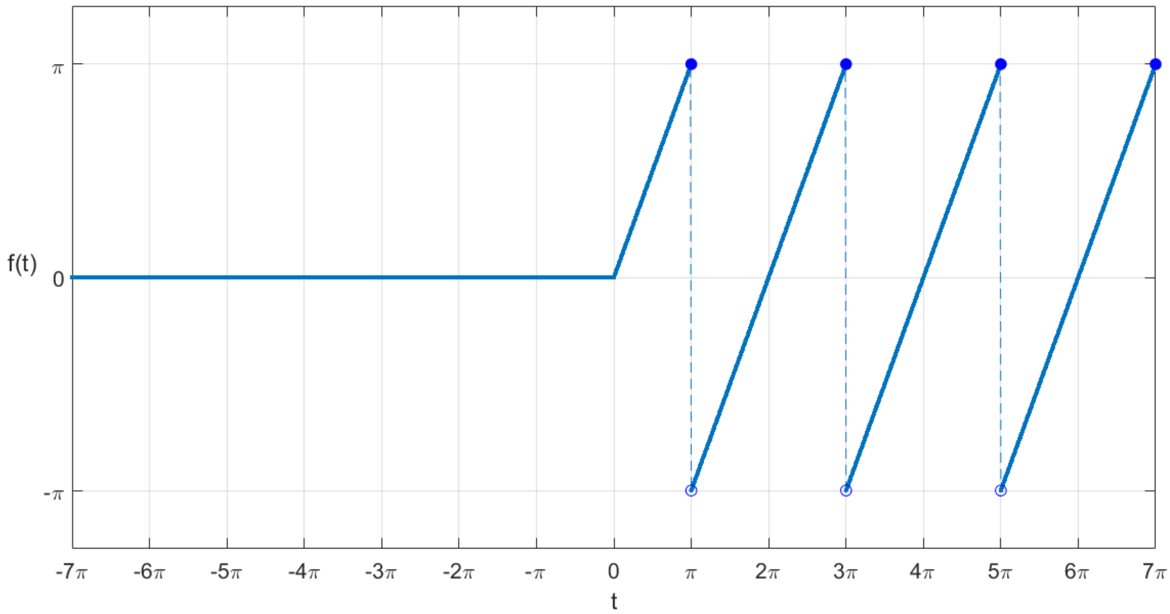
$F(w) = \dfrac{2sin(2w)}{w}$

$G(w) = 2 + e^{4w} + e^{-4w} = 2 + 2cos(4w)$

$F(w)G(w) = \left(\dfrac{2sin(2w)}{w}\right)(2 + 2cos(4w))$

Problem 5 [10 marks]

Evaluate the Laplace transform of the following function $f(t)$ and find its region of convergence. Notice that the function is periodic for $t \geq 0$ but is equal to 0 for $t < 0$.



Solution

$F(s) = \dfrac{F_T(s)}{1 - e^{-sT}}, \qquad T = 2\pi$

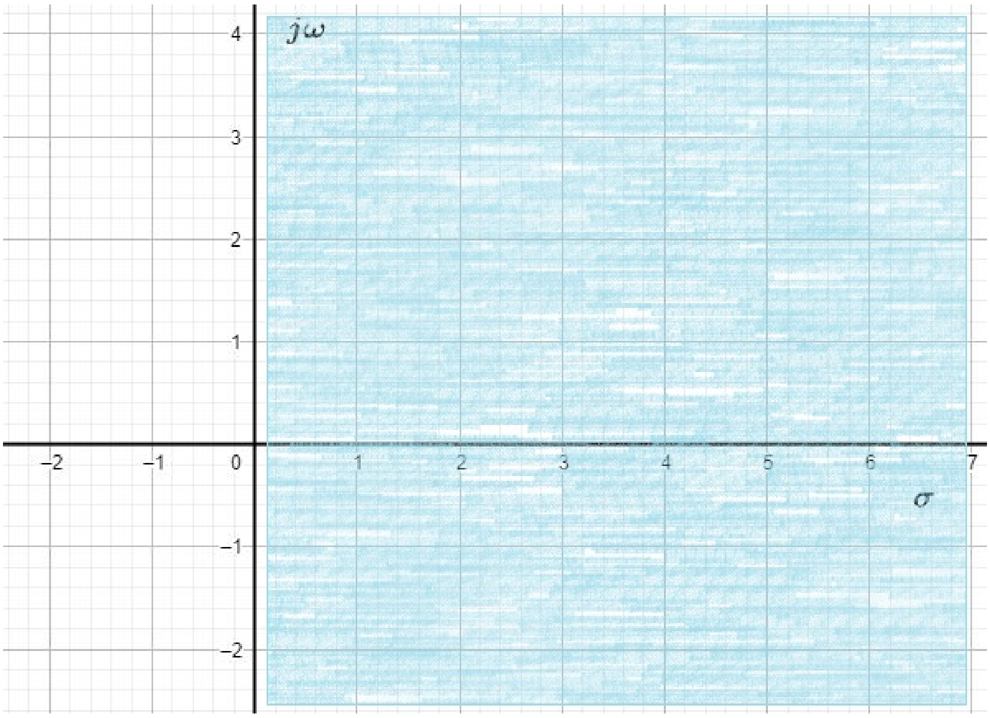
$F_T(s) = \int_0^t f(t) e^{-st}$

$F_T(s) = \int_0^\pi te^{-st}\ dt + \int_\pi^{2\pi} (t - 2\pi)e^{-st}\ dt$

$F_T(s) = \int_0^\pi te^{-st}\ dt - e^{2\pi s} \int_0^\pi te^{-st}\ dt$

$F_T(s) = \left(\dfrac{e^{\pi s} - e^{-\pi s}}{s^2}\right)(1 - e^{\pi s} + \pi s)$

$F_T(s) = \dfrac{e^{\pi s}}{s^2}\,(1 - e^{\pi} s + \pi s)$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