MT232: Differential Equations

Homework 6 Solution

Due Fri Dec 21, 9:00 AM *Fall 2018*

Problem 1

(a) Determine whether Laplace transform of given function exist? If yes then, find the range of values of s for which the Laplace transform $\mathcal{L}\{f(t)\}$ exists.

$$f(t) = t\sin t$$

A function $f(t)$ is said to be exponential type if it satisfies the following properties

(a) The function must be bounded above

$$0 \le |f(t)|e^{-st} < M$$

(b) Also this limit

$$\lim_{t\to\infty} |f(t)|e^{-st} = 0$$

Using triangular inequality and the fact that $e^{-st} > 0$, we get

$$|te^{-st}\sin t| \le |te^{-st}|, \quad \because 0 \le |\sin t| \le 1$$

As we know that at $t = 0$, $te^{-st} = 0$ and at $t \to \infty$ $te^{-st} \to 0$.
Therefore we need to check that between 0 and ∞, the below given condition hold up or not.

$$0 < |te^{-st}| < M$$

Now, Take derivative of te^{-st} to find maximum value of t.

$$t(-se^{-st}) + e^{-st} = 0$$

$$e^{-st}(1 - st) = 0$$

$$1 = st, \quad \implies t = \frac{1}{s}$$

Therefore, at this value of t, te^{-st} will be

$$\frac{1}{s}e^{-1} = \frac{1}{se}$$

$$\therefore M > \frac{1}{se}$$

which is bounded.
Now, to check limit

$$\lim_{t\to\infty} te^{-st}\sin t = \lim_{t\to\infty} \frac{t\sin t}{e^{st}}$$

Using Sandwich theorem

$$-1 < \sin(t) < 1$$

$$-t < t\sin t < t$$

$$-\frac{t}{e^{st}} < \frac{t\sin t}{e^{st}} < \frac{t}{e^{st}}$$

By applying limit

$$\lim_{t\to\infty} \frac{t}{e^{st}} = \frac{\infty}{\infty}$$

Using L'Hopital rule

$$\lim_{t\to\infty} \frac{1}{se^{st}} = 0$$

$$\lim_{t\to\infty} -\frac{t}{e^{st}} = 0$$

Hence,

$$\lim_{t\to\infty} \frac{t\sin t}{e^{st}} = 0$$

As both conditions satisfied. Therefore, it is exponential type.

(b) Consider the differential equation

$$y''(t) + 4y'(t) + 8y(t) = f(t)$$

where $f(t)$ is the input or forcing/driving function. Evaluate the following using Laplace Transform method and sketch their graphs.

(i) Impulse response (i.e. $f(t)$ is unit impulse, assuming zero initial conditions)

The differential equation is

$$y''(t) + 4y'(t) + 8y(t) = \delta(t)$$

where initial conditions for impulse response are defined as $y(0) = 0$ and $y'(0) = 0$.

Applying the Laplace transformation

$$\begin{aligned}
\mathcal{L}\{y''(t) + 4y'(t) + 8y(t)\} &= \mathcal{L}\{\delta(t)\}\\
\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y'(t)\} + 8\mathcal{L}\{y(t)\} &= \mathcal{L}\{\delta(t)\}\\
s^2Y(s) + 4sY(s) + 8Y(s) &= 1\\
(s^2 + 4s + 8)Y(s) &= 1\\
\implies Y(s) &= \frac{1}{(s^2 + 4s + 8)}\\
&= \frac{1}{(s^2 + 4s + 4) + 4}\\
&= \frac{1}{(s+2)^2 + 4}\\
&= \frac{1}{2}\frac{2}{(s+2)^2 + 2^2}\\
\implies y(t) &= \frac{1}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2 + 2^2}\right\}\\
&= \frac{1}{2}e^{-2t}\sin 2t
\end{aligned}$$

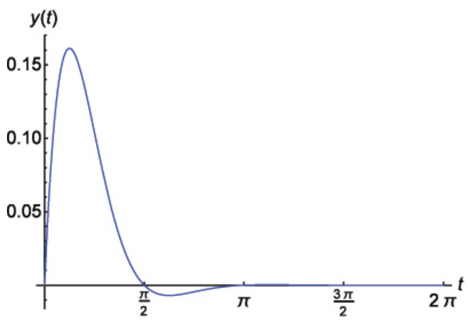


Figure 1: Impulse response

(ii) Step response (i.e. $f(t)$ is unit step, assuming zero initial conditions)

We can write the differential equation as

$$y''(t) + 4y'(t) + 8y(t) = h(t),$$

where initial conditions for step response are defined as $y(0) = 0$ and $y'(0) = 0$.

Applying the Laplace transformation

$$\begin{aligned}
\mathcal{L}\{y''(t) + 4y'(t) + 8y(t)\} &= \mathcal{L}\{h(t)\} \\
\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y'(t)\} + 8\mathcal{L}\{y(t)\} &= \mathcal{L}\{h(t)\} \\
s^2Y(s) + 4sY(s) + 8Y(s) &= \frac{1}{s} \\
(s^2 + 4s + 8)Y(s) &= \frac{1}{s} \\
\implies Y(s) &= \frac{1}{s((s+2)^2 + 4)}
\end{aligned}$$

Breaking into partial fraction of the form

$$\begin{aligned}
Y(s) &= \frac{A(s+2) + B}{(s+2)^2 + 4} + \frac{C}{s} \\
\text{We get } A &= -\frac{1}{8},\ B = -\frac{1}{4},\ C = \frac{1}{8} \\
\implies Y(s) &= -\frac{1}{8}\left(\frac{(s+2) + 2}{(s+2)^2 + 4}\right) + \frac{1}{8s} \\
Y(s) &= -\frac{1}{8}\left(\frac{s+2}{(s+2)^2 + 2^2}\right) - \frac{1}{8}\left(\frac{2}{(s+2)^2 + 2^2}\right) + \frac{1}{8s} \\
\implies y(t) &= -\frac{1}{8}e^{-2t}\cos 2t - \frac{1}{8}e^{-2t}\sin 2t + \frac{1}{8} \\
y(t) &= \frac{1}{8}(1 - e^{-2t}(\cos 2t + \sin 2t)) \\
\implies y(t) &= \frac{1}{8}\left(1 - \sqrt{2}e^{-2t}\cos\left(2t - \frac{\pi}{4}\right)\right)
\end{aligned}$$

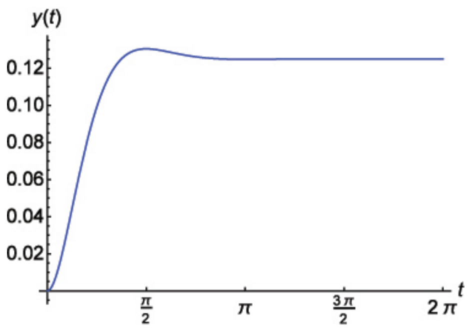


Figure 2: Step response

(iii) when $f(t) = e^{-2t}$, given $y(0) = 2$ and $y'(0) = 1$.

$$y''(t) + 4y'(t) + 8y(t) = e^{-2t}$$

Applying the Laplace transforms and plugging in the initial conditions

$$\begin{aligned}
\mathcal{L}\{y''(t) + 4y'(t) + 8y(t)\} &= \mathcal{L}\{e^{-2t}\} \\
(s^2Y(s) - sy(0) - y'(0)) + 4(sY(s) - y(0)) + 8Y(s) &= \frac{1}{s+2} \\
s^2Y(s) - 2s - 1 + 4sY(s) - 8 + 8Y(s) &= \frac{1}{s+2} \\
(s^2 + 4s + 8)Y(s) - (2s + 9) &= \frac{1}{s+2} \\
(s^2 + 4s + 8)Y(s) &= \frac{1}{s+2} + (2s + 9) \\
(s^2 + 4s + 8)Y(s) &= \frac{2s^2 + 13s + 19}{s+2} \\
\implies Y(s) &= \frac{2s^2 + 13s + 19}{((s+2)^2 + 4)(s+2)}
\end{aligned}$$

Breaking into partial fractions of the form

$$\begin{aligned}
Y(s) &= \frac{A(s+2) + B}{(s+2)^2 + 4} + \frac{C}{s+2} \\
\text{We get } A &= \frac{7}{4},\ B = 5,\ C = \frac{1}{4} \\
Y(s) &= \frac{7}{4}\left(\frac{s+2}{(s+2)^2 + 2^2}\right) + \frac{5}{(s+2)^2 + 2^2} + \frac{1}{4(s+2)} \\
y(t) &= \frac{7}{4}e^{-2t}\cos 2t + \frac{5}{2}\left(\frac{2}{(s+2)^2 + 2^2}\right) + \frac{1}{4}e^{-2t} \\
\implies y(t) &= \frac{7}{4}e^{-2t}\cos 2t + \frac{5}{2}e^{-2t}\sin 2t + \frac{1}{4}e^{-2t} \\
y(t) &= \frac{1}{4}e^{-2t}(1 + 7\cos 2t + 10\sin 2t)
\end{aligned}$$

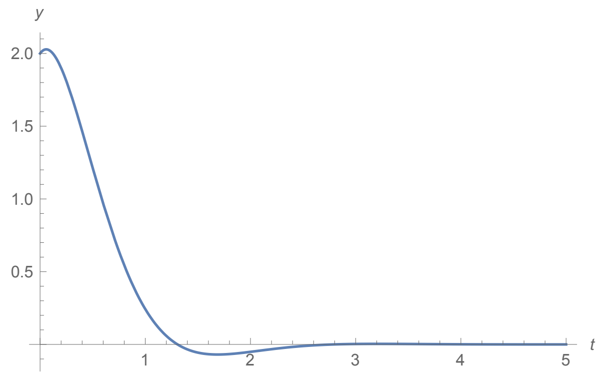


Figure 3: Response to input $f(t) = e^{-2t}$

Problem 2

Figure 4 shows a cart of mass m attached to a fixed spring with stiffness k and a dashpot with damping coefficient b. The cart is shown in its equilibrium position but it can have a displacement x towards right or left causing an extension or compression in the spring. Given that $m = 1$ kg and $k = 4$ Nm^{-1}.

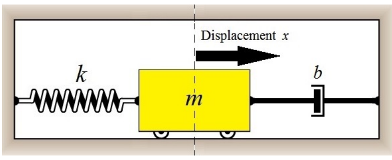


Figure 4: Spring mass system connected to a dashpot.

The cart is pulled 1 m to the right and released from rest.

(a) Write down a linear second order differential equation for this system, in terms of m, b and k.

$$m\ddot{x} + b\dot{x} + kx = 0 \tag{1}$$

(b) Write down the differential equation in terms of ζ and ω_o.

$$\ddot{x} + 2\zeta w_o \dot{x} + \omega_0^2 x = 0$$

(c) Without solving the differential equation, find the range of values of the damping coefficient b for which the cart oscillates but the amplitude of its oscillations decays to 13.5% within the first 2 seconds of release. What range of values of quality factor Q does this correspond to?

$$m\ddot{x} + b\dot{x} + kx = 0$$

Given $m = 1$, $k = 4$

$$\ddot{x} + b\dot{x} + 4x = 0$$

The characteristic equation

$$S(r) = 0$$
$$r^2 + br + 4 = 0$$
$$\implies r = \frac{-b \pm \sqrt{b^2 - 16}}{2}$$
$$= -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - 4}$$

Given that the cart oscillate, roots must be complex

$$\implies r = -\frac{b}{2} \pm i\sqrt{4 - \frac{b^2}{4}}$$

Because $e^{-\frac{b}{2}t}$ would scale the amplitude of oscillation so $-\frac{b}{2}$ is the term that causes decay of the oscillation. Now, if at $t = 0$ amplitude is 100%, and at $t = 2$, amplitude decays to 13.5%,

$$e^{-\frac{b}{2}(0)} = 1 \longrightarrow 100\%$$
$$e^{-\frac{b}{2}(2)} = 0.135 \longrightarrow 13.5\%$$

Solving the above equation for b,

$$-b = \ln 0.135$$
$$\implies b = 2$$

.

Consider $b = 2$ kgs^{-1} for rest of this problem. Now consider a new initial condition in which at $t = 0$, the cart passes its equilibrium position at a speed of 1 ms^{-1} towards right. At $t = 0$, a constant force of 4 N towards right is also applied to the cart and is maintained for all $t > 0$.

(d) Write down the differential equation of the system with an appropriate forcing function.
Now with system being driven by a force $F(t)$, the differential equation is given by

$$m\ddot{x} + b\dot{x} + kx = F(t) \tag{2}$$

Given $F = 4,\ b = 2,\ m = 1,\ k = 4$ and initial conditions $x(0) = 0,\ x'(0) = 1$, the differential equation becomes

$$\ddot{x} + 2\dot{x} + 4x = 4$$

(e) With these values of b and the force applied, the cart is expected to oscillate before approaching a position x_∞ as $t \to \infty$. Using a method of your choice, solve the differential equation and find the value of x_∞.
Applying Laplace transform and the initial conditions,

$$\mathcal{L}\{\ddot{x} + 2\dot{x} + 4x\} = \mathcal{L}\{4\}$$

$$\implies (s^2X(s) - sx(0) - x'(0)) + 2(sX(s) - x(0)) + 4X(s) = \frac{4}{s}$$

$$\begin{aligned} s^2X(s) - 1 + 2sX(s) + 4X(s) &= \frac{4}{s} \\ X(s)(s^2 + 2s + 4) - 1 &= \frac{4}{s} \\ X(s)(s^2 + 2s + 1 + 3) &= \frac{4}{s} + 1 \\ X(s)((s+1)^2 + 3) &= \frac{4+s}{s} \end{aligned}$$

$$\implies X(s) = \frac{s+4}{s((s+1)^2+3)}$$

Splitting into partial fraction of the form

$$\frac{s+4}{s((s+1)^2+3)} = \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2+3}$$

we get $A = 1,\ B = -1$ and $C = 0$,

$$\begin{aligned} \implies X(s) &= \frac{1}{s} - \frac{s+1}{(s+1)^2+3} \\ &= \frac{1}{s} - \frac{s+1}{(s+1)^2+(\sqrt{3})^2} \end{aligned}$$

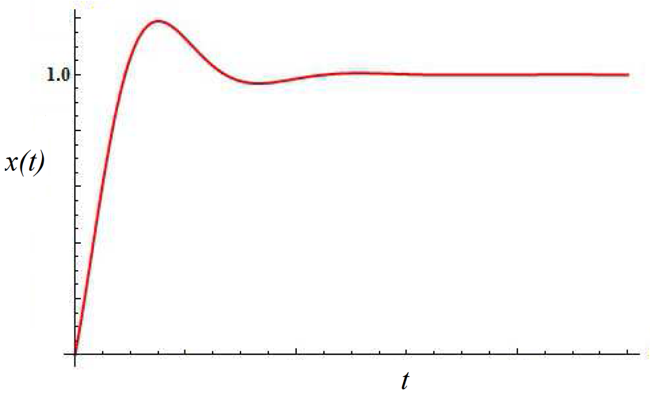
Now taking inverse Laplace transforms,

$$x(t) = 1 - e^{-t}\cos\sqrt{3}t$$

Finally,

$$x_\infty = \lim_{t\to\infty} x(t) = \lim_{t\to\infty}\left(1 - e^{-t}\cos\sqrt{3}t\right) = 1$$

(f) Sketch the graph of $x(t)$ for $t \geq 0$.



(g) Evaluate how far does the cart go past the point x_∞ for the first time during oscillation. (Look at your graph if you are having trouble understanding this part.)
From the graph, we can see that as the graph rises, it shoots over $x_\infty = 1$, attains a maximum value $x_{\max}$ and falls down. So, the overshoot $= x_{\max} - x_\infty$. Because $x_{\max}$ is the first stationary point of the graph, it can be found by putting $\dfrac{dx}{dt} = 0$,

$$\frac{dx}{dt} = e^{-t}\left(\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t\right) = 0$$

$$e^{-t}\left(\sqrt{3+1}\cos\left(\sqrt{3}t - \tan^{-1}\sqrt{3}\right)\right) = 0$$

$$2e^{-t}\cos\left(\sqrt{3}t - \frac{\pi}{3}\right) = 0$$

$$\implies \cos\left(\sqrt{3}t - \frac{\pi}{3}\right) = 0$$

$\cos\theta = 0$ for the first time at $\theta = \dfrac{\pi}{2}$. Hence,

$$\sqrt{3}t - \frac{\pi}{3} = \frac{\pi}{2}$$

$$\implies t = \frac{5\pi}{6\sqrt{3}}$$

For $t = \dfrac{5\pi}{6\sqrt{3}}$,

$$\begin{aligned} x_{\max} &= 1 - e^{-\frac{5\pi}{6\sqrt{3}}}\cos\left(\sqrt{3}\frac{5\pi}{6\sqrt{3}}\right) \\ &= 1.19 \\ \implies \text{overshoot} &= x_{\max} - x_\infty \\ &= 1.19 - 1 \\ &= 0.19 \end{aligned}$$

Problem 3

Suppose compartments A and B shown in Fig.3 are filled with fluids and are separated by a permeable membrane. The figure is a compartmental representation of the exterior and interior of a cell. Suppose, too, that a nutrient necessary for cell growth passes through the membrane.

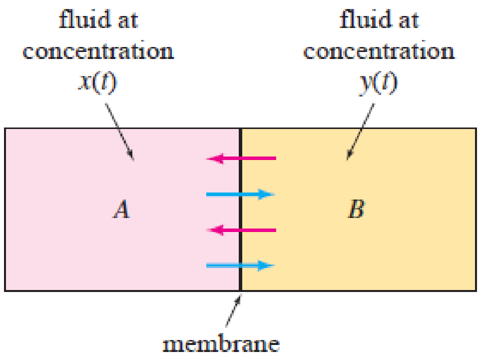


Figure 5: Nutrient flow through a membrane

A model for the concentrations $x(t)$ and $y(t)$ of the nutrient in compartments A and B, respectively, at time t is given by the linear system of differential equations

$$x'(t) = \frac{k}{V_A}(y - x)$$

$$y'(t) = \frac{k}{V_B}(x - y)$$

where V_A and V_B are the volumes of the compartments, and $k > 0$ is a permeability factor. Let $x(0) = x_0$ and $y(0) = y_0$ denote the initial concentrations of the nutrient. Solely on the basis of the equations in the system and the assumption $x_0 = 1$, $y_0 = 2$.

(a) Find its general solution for $V_A = V_B = 2\ \mathrm{m}^3$ and $k = 1$

The system can be converted into the following matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{A} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, we can write the system as

$$\mathbf{x}' = Ax$$

To find the eigenvalues of $\mathbf{A}$, the characteristic equation becomes

$$\lambda^2 - tr(\mathbf{A})\lambda + det(\mathbf{A}) = 0$$

$$\lambda^2 + \lambda = 0 \implies \lambda_{1,2} = 0, -1$$

Now to find the eigenvector v_1 for $\lambda_1 = 0$

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{1}{2}v_{11} + \frac{1}{2}v_{12} = 0 \implies v_{11} = v_{12}$$

Let $v_{12} = 1, \implies v_{11} = 1$. Hence, $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now, to find the eigenvector v_2 for $\lambda_2 = -1$

$$(A - \lambda_2 I)v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{1}{2}v_{21} + \frac{1}{2}v_{22} = 0 \implies v_{21} = -v_{22}$$

Let $v_{22} = 1, \implies v_{21} = -1$. Hence, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The general solution is

$$x(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(b) Sketch its general phase-portrait of the system, and show at least 8 trajectories in the phase-plane for $t \in (-\infty, \infty)$ starting from different initial conditions of your choice all over the plane (this is how we made those exotic sketches in class).

We discussed the case of system with two distinct real eigenvalues, repeated (nonzero) eigenvalue, purely imaginary eigenvalues and complex eigenvalues. But we did not discuss the case when one of the eigenvalues is zero. In fact, it is easy to see that this happen if and only if we have more than one equilibrium point. In this case, we will have a line of equilibrium points (the direction vector for this line is the eigenvector associated to the eigenvalue zero) .Here in this problem as you can see we have a zero eigenvalue and a nonzero eigenvalue. The phase portrait of such system will be as follow;

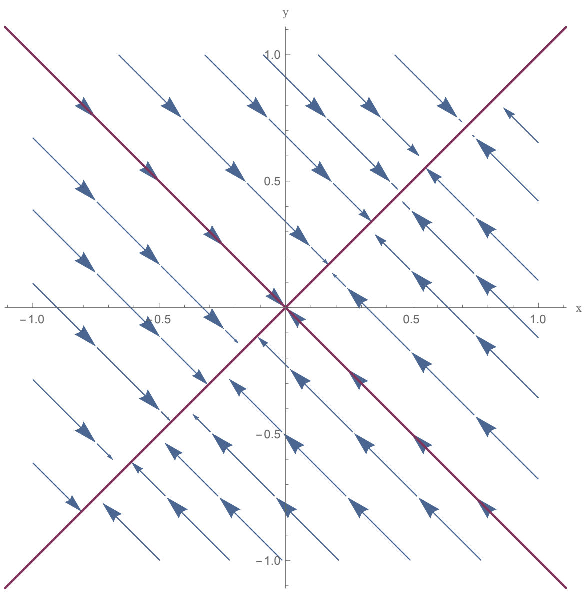


Figure 6: Phase Portrait

(c) Classify its equilibrium point as node, saddle, ellipse or spiral, and also comment on its stability.
In the phase portrait, every point on the brown line (associated with zero eigenvalue) is an equilibrium solution. This line is called attractive line of equilibria. So all the equilibrium points are stable.

(d) Now for $t \in (0, \infty)$, sketch the trajectory of its solution in the phase-plane that starts from the given initial condition at $t = 0$.

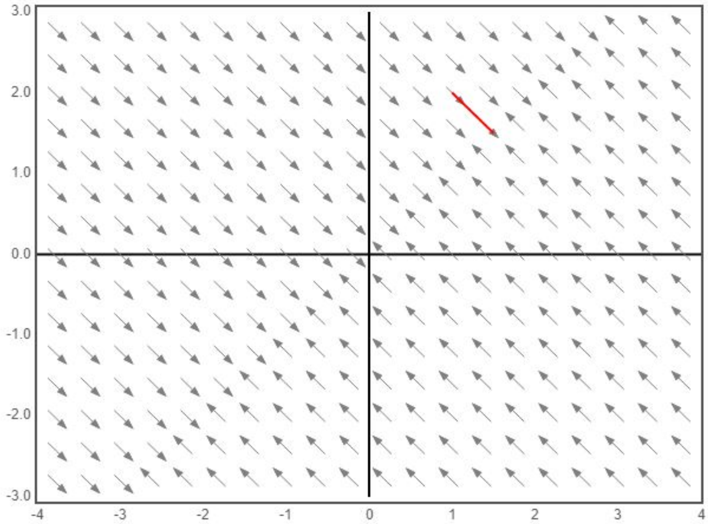


Figure 7: Phase Portrait

(e) Now use the given initial conditions to find its solution (i.e. evaluate c_1 and c_2).
Given the initial condition $x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

$$x(0) = c_1 \begin{bmatrix} 11 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$x(t) = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

(f) Initially which of the concentrations $x(t)$ or $y(t)$ of the nutrient changes fast.

$$x(t) = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$x(t) = \frac{1}{2}(3 - e^{-t})$$

$$y(t) = \frac{1}{2}(3 + e^{-t})$$

From the plots of x(t) and y(t), we can say that both the concentrations x(t) and y(t) of the nutrient changes at the same rate.

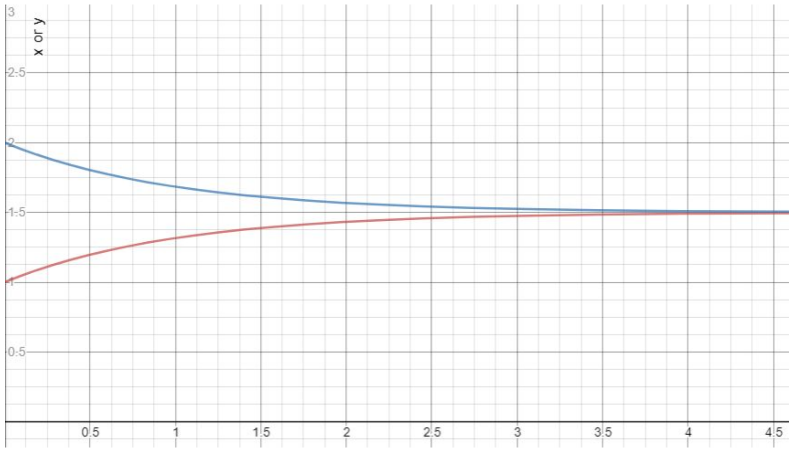


Figure 8: Plots of x(t) and y(t) against t

Problem 4

(a) A patient died during his treatment in a hospital. His relatives are putting the blame on a nurse who gave the patient an overdose of a drug Mathemine. But the nurse claims that although she did give a slight overdose but the amount she injected was still safe.

When Mathemine is injected into the muscle, it gradually flows into the bloodstream and mixes with the blood. Then over time, the body slowly gets rid of the drug through excretion. Let $x(t)$ be the amount of Mathemine in the blood at time t, and $y(t)$ be its amount in the muscle. The relationship between these quantities is given by the differential equations

$$x'(t) = -3x(t) + 2y(t)$$

$$y'(t) = x(t) - 2y(t)$$

Initially there was zero Mathemine in the patient's blood when the nurse injected 12 mg of Mathemine into his muscle. If the amount of Mathemine in the blood at a certain instance rises above 6 mg, it is lethal. Find out whether or not the patient died due to drug overdose and if the nurse is responsible for the death. (Hint: Find $x(t)$ by solving the system of differential equations and find out whether its value rises above 6 or not)

Solution

Given the initial condition $x(0) = 0$, $y(0) = 12$ and condition for death $x(t) > 6$ The system can be converted into the following matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$, we can write the system as

$$\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$$

To find the eigenvalues of $\mathbf{A}$, the characteristic equation becomes

$$\lambda^2 - tr(\mathbf{A})\lambda + det(\mathbf{A}) = 0$$

$$\lambda^2 + 5\lambda + 4 = 0$$

$$(\lambda + 4)(\lambda + 1) = 0$$

$$\implies \lambda_1 = -1, \quad \lambda_2 = -4$$

Now to find the eigenvector v_1 for $\lambda_1 = -1$

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_{11} - v_{12} = 0 \implies v_{11} = v_{12}$$

Let $v_{11} = 1, \implies v_{12} = 1$. Hence, $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now for eigenvector v_2 for $\lambda_2 = -4$,

$$(A - \lambda_2 I)v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$v_{21} + 2v_{22} = 0 \implies v_{21} = -2v_{22}$$

Let $v_{22} = -7, \implies v_{21} = 8$. Hence, $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

The general solution is

$$x(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Given the initial condition $x(0) = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$,

$$x(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$$

$$\implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 12 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 12 \end{bmatrix}$$

$$\implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$x(t) = 8e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4e^{-4t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\implies x(t) = 8e^{-t} - 8e^{-4t}$$

From the equation of $x(t)$, we can deduce that it starts from 0, rises to some maximum value and then decays back to 0 over time. To check whether $x(t) > 6$ for any value of $t > 0$, we need to find it maximum value.

$$\frac{dx}{dt} = -8e^{-t} + 32e^{-4t} = 0$$

$$-8e^{-t}(1 - 4e^{-3t}) = 0$$

$$\implies 4e^{-3t} = 1$$

$$\implies t = -\frac{1}{3}\ln\frac{1}{4} \approx 0.4621$$

At $t = 0.4621$, $x(t) \approx 3.78$ mg.

So, the amount of mathemine in patient's bloodstream did not rise above 6 mg for a value of $t > 0$. So, the nurse is not responsible for the patient's death.

(b) In an $n \times n$ linear system of differential equations with constant coefficients

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),$$

prove that if matrix $\mathbf{A}$ is non-singular (invertible), any value of velocity vector is achievable at some point in the phase plane.

Solution

Consider an arbitrary velocity vector v. If we are able to show that for any value of v, there exists a corresponding point x in phase-plane, we will be done. So for any given velocity vector v,

$$v = \mathbf{x}' = \mathbf{A}\mathbf{x}$$

$$\implies x = A^{-1}v$$

Now, because A is invertible, we can always find a vector x corresponding to any given v. ■

Problems 5

Tank T_1 and T_2 in Fig. 3 contain initially 100 gallons of water each. In T_1 the water is pure, whereas 180 kg of fertilizer are dissolved in T_2. By circulating liquid at a rate of 2 gal/min and stirring (to keep the mixture uniform) the amounts of fertilizer $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t.

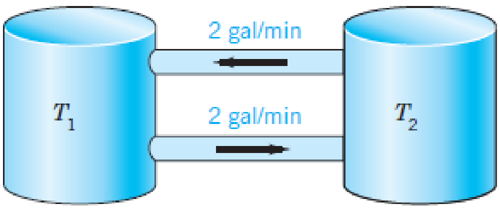


Figure 9: System of tanks

(a) Write the differential equations that describes the above mathematical model.

As for a single tank, the time rate of change y'(t) of y(t) equals inflow minus outflow. Similarly for tank T2.

$$y_1' = Inflow/min - Outflow/min = \frac{2}{100}y_2 - \frac{2}{100}y_1$$

$$y_2' = Inflow/min - Outflow/min = \frac{2}{100}y_1 - \frac{2}{100}y_2$$

Hence the mathematical model of our mixture problem is

$$y_1' = -0.02y_1 + 0.02y_1$$

$$y_2' = 0.02y_1 - 0.02y_2$$

(b) Find the general solution of the above mathematical model.
The system can be converted into the following matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{A} = \begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix}$, we can write the system as

$$\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$$

To find the eigenvalues of $\mathbf{A}$, the characteristic equation becomes

$$\lambda^2 - tr(\mathbf{A})\lambda + det(\mathbf{A}) = 0$$

$$\lambda^2 + 0.04\lambda + 0 = 0$$

$$\lambda(\lambda + 0.04) = 0$$

$$\implies \lambda_1 = 0, \quad \lambda_2 = -0.04$$

Now to find the eigenvector v_1 for $\lambda_1 = 0$

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-0.02v_{11} + 0.02v_{12} = 0 \implies v_{11} = v_{12}$$

Let $v_{11} = 1, \implies v_{12} = 1$. Hence, $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now for eigenvector v_2 for $\lambda_2 = -0.04$,

$$(A - \lambda_2 I)v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.02 & 0.02 \\ 0.02 & 0.02 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$0.02v_{21} + 0.02v_{22} = 0 \implies v_{21} = -v_{22}$$

Let $v_{22} = -1, \implies v_{21} = 1$. Hence, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The general solution is

$$x(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-0.04t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(c) Sketch the general phase portrait of the system.

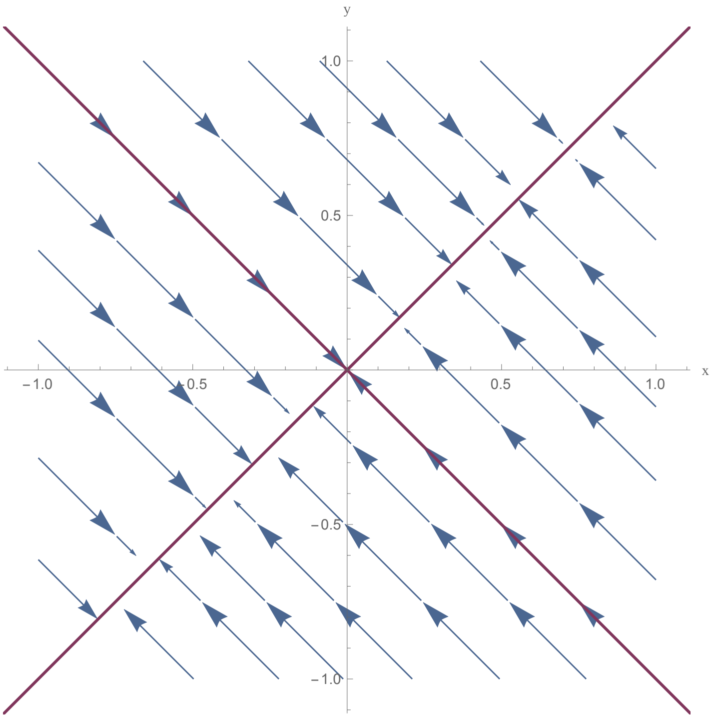


Figure 10: Phase Portrait

(d) How long should we let the liquid circulate so that T_1 will contain at least half as much fertilizer as there will be left in T_2?

Given the initial condition $x(0) = \begin{bmatrix} 0 \\ 180 \end{bmatrix}$,

$$x(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 180 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 180 \end{bmatrix}$$

$$\implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 180 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 180 \end{bmatrix}$$

$$\implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 90 \\ -90 \end{bmatrix}$$

$$x(t) = 90 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 90e^{-0.04t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\implies y_1(t) = 90 - 90e^{-0.04t}$$

$$\implies y_2(t) = 90 + 90e^{-0.04t}$$

Figure 10 shows the exponential increase of y_1 and the exponential decrease of y_2 to the common limit 90kg. T_1 contains half the fertilizer amount of T_2 if it contains $\frac{1}{3}$ of the total amount, that is, 60kg. Thus

$$y_1 = 90 - 90e^{-0.04t} = 60 \implies e^{-0.04t} = \frac{1}{3} \implies t = \frac{\ln 3}{0.04} = 27.5$$

Hence the fluid should circulate for at least about half an hour.

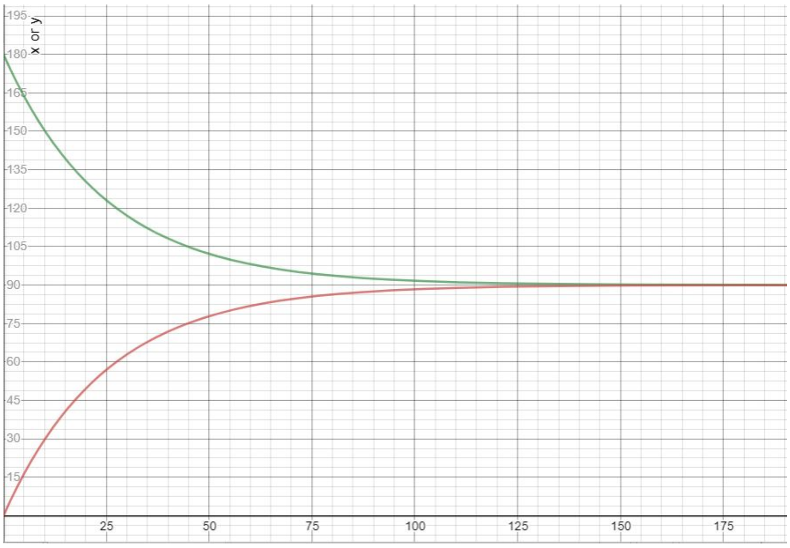


Figure 11: Plots of $y_1(t)$ and $y_2(t)$ against t

Problem 6

Consider the differential equation

$$y''(t) + 7y'(t) + 12y(t) = te^{-3t},$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

(a) Convert the differential equation into a linear system of first order differential equations.

Let, $y = y_1$ and $y' = y_2$

$y' = y_1' = y_2$ and $y'' = y_2'$

$y_2' + 7y_2 + 12y_1 = te^{-3t} \implies y_2' = -7y_2 - 12y_1 + te^{-3t}$

The system can be converted into the following matrix form

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ te^{-3t} \end{bmatrix}$$

Letting $\mathbf{x} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -12 & -7 \end{bmatrix}$, and $\mathbf{f} = \begin{bmatrix} 0 \\ te^{-3t} \end{bmatrix}$ we can write the system as

$$\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}$$

(b) Find the complementary solution of the system.

To find the complementary solution of the system we will put f = 0 and find the eigenvalues of $\mathbf{A}$, the characteristic equation becomes

$$\lambda^2 - tr(\mathbf{A})\lambda + det(\mathbf{A}) = 0$$

$$\lambda^2 + 7\lambda + 12 = 0$$

$$(\lambda + 3)(\lambda + 4) = 0$$

$$\implies \lambda_1 = -3, \quad \lambda_2 = -4$$

Now to find the eigenvector v_1 for $\lambda_1 = -3$

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ -12 & -4 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3v_{11} + v_{12} = 0 \implies v_{12} = -3v_{13}$$

Let $v_{11} = 1, \implies v_{12} = -3$. Hence, $v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Now for eigenvector v_2 for $\lambda_2 = -4$,

$$(A - \lambda_2 I)v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ -12 & -3 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$4v_{21} + v_{22} = 0 \implies v_{22} = -4v_{21}$$

Let $v_{21} = 1, \implies v_{21} = -4$. Hence, $v_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

The general solution is

$$x(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

(c) Find the particular solution of the system using variation of parameters.
Particular solution of the system can be found using variation of parameters as below;

$$X_p = \bar{X} \int \bar{X}^{-1} f dt$$

$$\bar{\mathbf{X}} = \begin{bmatrix} e^{-3t} & e^{-4t} \\ -3e^{-3t} & -4e^{-4t} \end{bmatrix}$$

$$\bar{\mathbf{X}}^{-1} = \begin{bmatrix} 4e^{3t} & e^{3t} \\ -3e^{4t} & -e^{4t} \end{bmatrix}$$

$$X_p = \bar{X} \int \bar{X}^{-1} f dt$$

$$X_p = \begin{bmatrix} e^{-3t} & e^{-4t} \\ -3e^{-3t} & -4e^{-4t} \end{bmatrix} \int \begin{bmatrix} 4e^{3t} & e^{3t} \\ -3e^{4t} & -e^{4t} \end{bmatrix} \begin{bmatrix} 0 \\ te^{-3t} \end{bmatrix} dt$$

$$X_p = \begin{bmatrix} e^{-3t}(\frac{t^2}{2} + t - 1) \\ e^{-3t}(\frac{-3t^2}{2} - 4t + 4) \end{bmatrix}$$

(d) Determine the solution $y(t)$ of the differential equation.

$$X = X_c + X_p$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ -4 \end{bmatrix} + \begin{bmatrix} e^{-3t}(\frac{t^2}{2} + t - 1) \\ e^{-3t}(\frac{-3t^2}{2} - 4t + 4) \end{bmatrix}$$

$$y(t) = y_1 = c_1 e^{-3t} + c_2 e^{-4t} + e^{-3t}(\frac{t^2}{2} + t - 1)$$

Given the initial conditions $y(0) = 0$ and $y'(0) = 1$.

$$y(0) = c_1 + c_2 - 1 = 0 \implies c_1 + c_2 = 1 Eq(1)$$

$$y'(0) = -3c_1 - 4c_2 + 4 = 1 \implies 3c_1 + 4c_2 = 3 Eq(2)$$

Solving Eq1 and Eq2 simultaneously, we get
$c_1 = 1$ and $c_2 = 0$ so the final solution is

$$y(t) = e^{-3t} + e^{-3t}(\frac{t^2}{2} + t - 1) \implies te^{-3t}(\frac{t}{2} + 1)$$

THE END
