MT232: Differential Equations

Homework 5 Solution

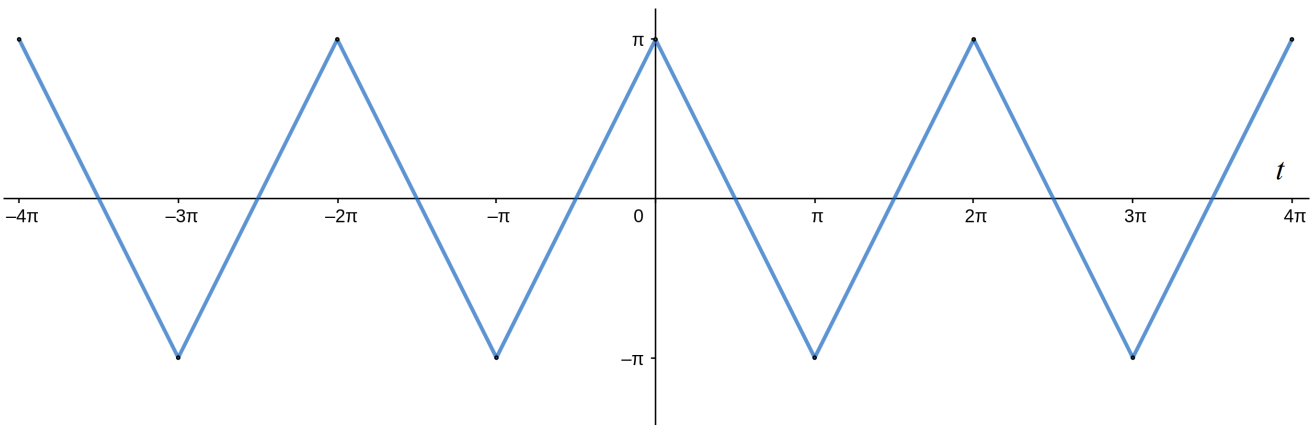
Fall 2018

Tips to avoid plagiarism

- Do not copy the solutions of your classmates.
- Your are encouraged to discuss the problems with your classmates in whatever way you like but make sure to REPRODUCE YOUR OWN SOLUTIONS in what you submit for grading.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1

(a) Consider the triangle wave given below.



Periodic Function $f_1(t)$

(i) Determine its time period T, frequency f and angular frequency ω.

$$T = 2\pi$$

As we know that

$$f = \frac{1}{T} = \frac{1}{2\pi}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi}, \qquad \implies \omega = 1 rads^{-1}$$

(ii) Write down its expression in the intervals $-\frac{T}{2} < t \leq 0$ and $0 < t \leq \frac{T}{2}$.
In the interval $-\pi < t \leq 0$, two points on the line are $(-\pi, -\pi)$ and $(0, \pi)$. Slope of the line is 2 in this interval and y-intercept is π. So, the equation of the line is $2t + \pi$ from $-\pi < t \leq 0$. In the interval $0 < t \leq \pi$, two points on the line are $(-\pi, \pi)$ and $(0, \pi)$. Slope of the line is -2 in this interval and y-intercept is π. So, the equation of the line is $-2t + \pi$ from $-\pi < t \leq 0$.

$$f_1(t) = \begin{cases} 2t + \pi & -\pi < t \leq 0 \\ -2t + \pi & 0 < t \leq \pi \end{cases}$$

(iii) Is this an even function or an odd function? Explain.
It is an Even function because $f_1(t) = f_1(-t)$.

(iv) Find its Fourier series.
Only a_n exist as the function is even.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt$$
$$a_n = 2\left(\frac{1}{\pi}\right)\int_0^{\pi} (-2t+\pi)\cos(nt)dt$$
$$a_n = \frac{2}{\pi}\left(-2\int_0^{\pi} t\cos(nt)dt + \int_0^{\pi} \pi\cos(nt)dt\right)$$
$$a_n = \frac{2}{\pi}\left(-2\left(\frac{t}{n}\sin(nt)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nt)dt\right) + \frac{\pi}{n}\sin(nt)\Big|_0^{\pi}\right)$$
$$a_n = \frac{2}{\pi}\left(-2\left(0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right) + 0\right), \quad \because \sin n\pi = 0$$
$$a_n = \frac{4}{\pi n^2}(1-\cos n\pi)$$
$$a_n = \frac{4}{\pi n^2}(1+(-1)^{n+1})$$
$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$
$$a_o = \frac{2}{\pi}\int_0^{\pi} (-2t+\pi)dt$$
$$a_o = \frac{2}{\pi}(-t^2+\pi t)\Big|_0^{\pi} = 0$$

Therefore, Fourier series will be

$$f(t) = \sum_{n=1}^{\infty} a_n\cos(nt)$$
$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}(1+(-1)^{n+1})\cos(nt)$$

(v) Evaluate the Fourier series coefficients from $n = 0$ to $n = 9$.

$$a_0 = 0$$
$$a_1 = \frac{8}{\pi}$$
$$a_2 = 0$$
$$a_3 = \frac{8}{9\pi}$$
$$a_4 = 0$$
$$a_5 = \frac{8}{25\pi}$$
$$a_6 = 0$$
$$a_7 = \frac{8}{49\pi}$$
$$a_8 = 0$$
$$a_9 = \frac{8}{81\pi}$$

(vi) Find the particular solution of the following differential equation, if $f(t) = f_1(t)$ is the input function of the following differential equation.

$$y'' + 25y = f(t)$$

$$y'' + 25y = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}(1 + (-1)^{n+1})\cos(nt)$$

We first calculate the solution of the following equation:

$$y'' + 25y = \cos(nt)$$

$$S(r) = r^2 + 25$$

$$\alpha = in$$

The particular solution for the above equation can be found as following:

$$y_p = \Re\left(\frac{\tilde{y_p}}{S(\alpha)}\right) = \Re\left(\frac{\tilde{e^{int}}}{\alpha^2 + 25}\right)$$

For $n \neq 5$,

$$y_p = \frac{\cos nt}{25 - n^2}$$

Now using the linearity property, y_p for $\cos nt$ can be used to find the particular solution of $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}(1 + (-1)^{n+1})\cos(nt)$. The solution comes out to be:

$$y_{p1} = \frac{4}{\pi}\sum_{\substack{n=1\\n\neq 5}}^{\infty}\frac{1}{n^2}(1 + (-1)^{n+1})\frac{\cos(nt)}{25 - n^2}$$

Now, we will evaluate y_{p2} at $n = 5$.

$$\alpha = i5, B = \frac{8}{25\pi}$$

$$\begin{aligned}
\tilde{y}_{p2} &= B\frac{te^{i5t}}{S'(\alpha)} \\
\tilde{y}_{p2} &= \frac{8}{25\pi}\frac{te^{i5t}}{10i} \\
\tilde{y}_{p2} &= \frac{8}{10(25\pi)}\frac{te^{i5t}}{i}\left(\frac{-i}{-i}\right) \\
\tilde{y}_{p2} &= \frac{4}{125\pi}t(\cos 5t + i\sin 5t)(-i) \\
y_{p2} &= \Re\{\tilde{y}_{p2}\} = \frac{4}{125\pi^2}t\sin 5t
\end{aligned}$$

$$\begin{aligned}
y_p &= y_{p1} + y_{p2} \\
y_p &= \frac{4}{\pi}\sum_{\substack{n=1\\n\neq 5}}^{\infty}\frac{1}{n^2}(1 + (-1)^{n+1})\frac{\cos(nt)}{25 - n^2} + \frac{4}{125\pi^2}t\sin 5t
\end{aligned}$$

(vii) What is the resonant frequency of the system in (vi)?

$$\omega_r = \omega_o = \sqrt{25} = 5$$

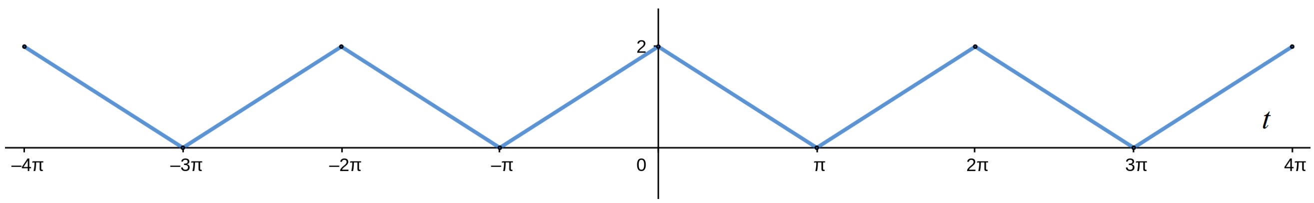
(viii) Would any frequency component of $f_1(t)$ cause resonance in the system? If yes, express that resonant component of the output in terms of t, demonstrating that its amplitude is increasing indefinitely.

Yes the component at $n = 5$ will cause resonance in system. The output component corresponding this frequency is y_{p2}

$$y_{p2} = \frac{4}{125\pi^2} t \sin 5t$$

As, there is t in the numerator which will keep the amplitude of oscillation increasing.

(b) Now consider the triangle wave given below.



Periodic Function $f_2(t)$

Repeat all the steps (i)-(viii) in (a) for the triangle wave given in this figure. Use $f(t) = f_2(t)$ for this part. In part (iv), use the Fourier series of $f_1(t)$ to compute the Fourier series of $f_2(t)$, i.e. do not use the Fourier coefficient formulas to evaluate the Fourier series.

(i) Determine its time period T, frequency f and angular frequency ω.

$$T = 2\pi$$

As we know that

$$f = \frac{1}{T} = \frac{1}{2\pi}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi}, \qquad \implies \omega = 1 rads^{-1}$$

(ii) Write down its expression in the intervals $-\frac{T}{2} < t \leq 0$ and $0 < t \leq \frac{T}{2}$.

$$f_2(t) = \begin{cases} \frac{2t}{\pi} + 2 & -\pi < t \leq 0 \\ -\frac{2t}{\pi} + 2 & 0 < t \leq \pi \end{cases}$$

(iii) Is this an even function or an odd function? Explain.

It is an Even function. So only a_n exists.

(iv) Find its Fourier series.

First, we π to $f_1(t)$ with -1 which will shift the graph on by π. Then scaled it with $\frac{1}{\pi}$.

$$f_2(t) = \frac{1}{\pi}(f_1(t) + \pi)$$

As, we need to find Fourier series of $f_2(t)$. Therefore,

$$f_2(t) = \frac{1}{\pi}(f_1(t) + \pi)$$

We have already computed Fourier series of $f_1(t)$ which is

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}(1+(-1)^{n+1})\cos(n\pi)$$

Therefore,

$$g(t) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(1+(-1)^{n+1})\cos(n\pi)$$

(v) Evaluate the Fourier series coefficients from $n = 0$ to $n = 9$.

$$a_0 = 2$$
$$a_1 = 1 + \frac{8}{\pi^2}$$
$$a_2 = 1$$
$$a_3 = 1 + \frac{8}{9\pi^2}$$
$$a_4 = 1$$
$$a_5 = 1 + \frac{8}{25\pi^2}$$
$$a_6 = 1$$
$$a_7 = 1 + \frac{8}{49\pi^2}$$
$$a_8 = 1$$
$$a_9 = 1 + \frac{8}{81\pi^2}$$

(vi) Find the particular solution of the following differential equation, if $f(t) = f_1(t)$ is the input function of the following differential equation.

$$y'' + 25y = f(t)$$

In this case $f(t) = f_2(t)$

$$y'' + 25y = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(1+(-1)^{n+1})\cos(n\pi)$$

$$P(r) = r^2 + 25$$

The new function $f_2(t)$ can be written as following using the linearity property:

$$f_2(t) = g_1(t) + g_2(t)$$

Where,

$$g_1(t) = 1, \quad g_2(t) = -\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(1+(-1)^{n+1})\cos(n\pi)$$

For $g_1(t)$

$$y_{p1} = \frac{Be^{\alpha t}}{S(\alpha)} = \frac{e^0}{25}, \quad \Longrightarrow y_{p1}(t) = \frac{1}{25}$$

y_{p2} for $g_2(t)$ is y_p computed for $f_1(t)$ multiplied by $\frac{1}{\pi}$. Therefore,

$$y_{p2} = \frac{4}{\pi^2} \sum_{\substack{n=1 \\ n \neq 5}}^{\infty} \frac{1}{n^2}(1 + (-1)^{n+1}) \frac{\cos(nt)}{25 - n^2} + \frac{4}{125\pi^2} t \sin 5t$$

Now, y_p can be written as the sum of y_{p1} and y_{p2} due to linearity property.

$$y_p = y_{p1} + y_{p2}$$

$$y_p = \frac{1}{25} + \left(\frac{4}{\pi^2} \sum_{\substack{n=1 \\ n \neq 5}}^{\infty} \frac{1}{n^2}(1 + (-1)^{n+1}) \frac{\cos(nt)}{25 - n^2} + \frac{4}{125\pi^2} t \sin 5t \right)$$

$$\therefore y_p = \frac{1}{25} + \frac{4}{\pi^2} \sum_{\substack{n=1 \\ n \neq 5}}^{\infty} \frac{1}{n^2}(1 + (-1)^{n+1}) \frac{\cos(nt)}{25 - n^2} + \frac{4}{125\pi^2} t \sin 5t$$

(vii) What is the resonant frequency of the system in (vi)?

$$\omega_r = \omega_o = \sqrt{25} = 5$$

(viii) Would any frequency component of $f_1(t)$ cause resonance in the system? If yes, express that resonant component of the output in terms of t, demonstrating that its amplitude is increasing indefinitely.

Yes the component at $n = 5$ will cause resonance in system. The output component corresponding this frequency is y_{p2}

$$y_{p2} = \frac{4}{125\pi^2} t \sin 5t$$

As, there is t in the numerator which will keep the amplitude of oscillation increasing.

Problems 2

Solve the following differential equations by any method of your choice (complexification, undetermined coefficients, variation of parameters or any combination thereof).
Hint: Concepts of linearity and superposition will be helpful.

(a) $\frac{1}{4}y'' + y' + y = x^2 - 2x$

$$y'' + 4y' + 4y = 4x^2 - 8x$$
$$r^2 + 4r + 4 = 0$$
$$(r+2)^2 = 0$$
$$r = -2$$
$$\text{As the roots are repeated, } y_c = (c_1 + c_2 x)e^{-2x}$$
$$\text{Let } y_p = Ax^2 + Bx + C$$
$$y_p' = 2Ax + B$$
$$y_p'' = 2A$$
$$\text{Putting the values of } y_p,\ y_p' \text{ and } y_p'' \text{ in the given equation}$$
$$2A + 4(2Ax + b) + 4(Ax + B) + 4(Ax^2 + Bx + C) = 4x^2 - 8x$$
$$4Ax^2 + 8Ax + 4Bx + 2A + 4B + 4C = 4x^2 - 8x$$
$$\text{Comparing coefficients, } A = 1,$$
$$B = -4,$$
$$C = \frac{7}{2}$$
$$\text{Putting the values of A,B and C in } y_p$$
$$y_p = x^2 - 4x + \frac{7}{2}$$
$$y = y_c + y_p$$
$$y = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}$$

(b) $y'' - y' - 6y = e^{-2x}\sin 2x + 12x$

$$y'' - y' - 6y = e^{-2x}\sin 2x + 12x$$
$$r^2 - r - 6 = 0$$
$$r = -2,\ 3$$
$$y_c = c_1e^{-2x} + c_2e^{3x}$$

Considering only $y'' - y' - 6y = 12x$

$$y_{p1} = Ax + B$$
$$y'_{p1} = A$$
$$y''_{p1} = 0$$

Putting the values of $y_{p1},\ y'_{p1}$ and y''_{p1} in the given equation

$$0 - A - 6(Ax + B) = 12x$$
$$-6Ax - A - 6B = 12x$$

Comparing coefficients, $A = -\frac{1}{2},$

$$B = \frac{1}{12},$$

Putting the values of A,B in y_{p1}

$$y_{p1} = -\frac{x}{2} + \frac{1}{12}$$

Considerig only $y'' - y' - 6y = e^{-2x}\sin 2x$

$$\tilde{y}_{p2} = e^{-2x}[\cos 2x + i\sin 2x]$$
$$y_{p2} = \Im\{\tilde{y}_{p2}\}$$
$$\alpha = 2i - 2,\ B = 1$$
$$\tilde{y}_{p2} = B\frac{e^{\alpha x}}{S(\alpha)}$$
$$\tilde{y}_{p2} = \frac{e^{(2i-2)x}}{-4 - 6i}$$
$$\tilde{y}_{p2} = \frac{e^{-2x}(-4 + 6i)(\cos 2x + i\sin 2x)}{52}$$
$$y_{p2} = \Im\{\tilde{y}_{p2}\} = \frac{e^{-2x}}{52}(6\cos 2x - 4\sin 2x)$$

Using superposition principle,

$$y_p = y_{p1} + y_{p2}$$
$$y_p = -\frac{x}{2} + \frac{1}{12} + \frac{e^{-2x}}{52}(6\cos 2x - 4\sin 2x)$$
$$y = y_c + y_p$$
$$y = c_1e^{-2x} + c_2e^{3x} - \frac{x}{2} + \frac{1}{12} + \frac{e^{-2x}}{52}(6\cos 2x - 4\sin 2x)$$

(c) $y'' + 3y' + 2y = \dfrac{1}{1+e^x}$

$$\begin{aligned}
y'' + 3y' + 2y &= \frac{1}{1+e^x} \\
r^2 + 3r + 2 &= 0 \\
r &= -1, \ -2 \\
y_c &= c_1 e^{-x} + c_2 e^{-2x} \\
\text{So, } y_1 &= e^{-x} \\
y_2 &= e^{-2x}
\end{aligned}$$

Finding the Wroskian

$$\begin{aligned}
W &= \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-x} \end{vmatrix} \\
&= -2e^{-3x} + e^{-3x} = -e^{-3x} \\
u_1' &= -\frac{y_2 f(x)}{W} = \frac{e^x}{1+e^x} \\
u_2' &= \frac{y_1 f(x)}{W} = -\frac{e^{2x}}{1+e^x} \\
u_1 &= \int \frac{e^x}{1+e^x} dt \\
u_1 &= \ln|1+e^x| \\
u_2 &= \int \frac{e^{2x}}{1+e^x} dt \\
u_2 &= -(e^x - \ln|1+e^x|) \\
y_p &= y_1 u_1 + y_2 u_2 \\
y_p &= e^{-x}\ln|1+e^x| - e^{-2x}(e^x - \ln|1+e^x|) \\
y_p &= e^{-x}\ln|1+e^x| - e^{-x} + e^{-2x}\ln|1+e^x|) \\
y_p &= (e^{-x} + e^{-2x})\ln|1+e^x| - e^{-x} \\
y &= y_c + y_p \\
y &= c_1 e^{-x} + c_2 e^{-2x} + (e^{-x} + e^{-2x})\ln|1+e^x| - e^{-x} \\
y &= (c_1 - 1)e^{-x} + c_2 e^{-2x} + (e^{-x} + e^{-2x})\ln|1+e^x| \\
y &= ce^{-x} + c_2 e^{-2x} + (e^{-x} + e^{-2x})\ln|1+e^x| \quad (c_1 - 1 = c)
\end{aligned}$$

(d) $2y'' + 18y = 6\tan 3t$

$$y'' + 9y = 3tan(3t)$$
$$r^2 + 9 = 0$$
$$r = +3\imath, \quad -3\imath$$
$$y_c = c_1 \cos 3t + c_2 \sin 3t$$
$$\text{So, } y_1 = \cos 3t$$
$$y_2 = \sin 3t$$

Finding the Wroskian

$$W = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix}$$
$$= 3\cos^2 3t + 3\sin^2 3t = 3$$
$$u_1' = -\frac{y_2 f(x)}{W} = \frac{3\sin 3t \tan 3t}{3}$$
$$u_2' = \frac{y_1 f(x)}{W} = \frac{3\cos 3t \tan 3t}{3}$$
$$u_1 = -\int \frac{3\sin 3t \tan 3t}{3} dt$$
$$u_1 = -\int \frac{\sin^2 3t}{\cos 3t} dt$$
$$u_1 = -\int \frac{1-\cos^2 3t}{\cos 3t} dt$$
$$u_1 = -\int (\sec 3t - \cos 3t) dt$$
$$u_1 = -\frac{1}{3}(\ln|\sec 3t + \tan 3t| - \sin 3t)$$
$$u_2 = \int \frac{3\cos 3t \tan 3t}{3} dt$$
$$u_2 = \int \sin 3t dt$$
$$u_2 = -\frac{\cos 3t}{3}$$
$$y_p = y_1 u_1 + y_2 u_2$$
$$y_p = \cos 3t\left(-\frac{1}{3}(\ln|\sec 3t + \tan 3t| - \sin 3t)\right) + \sin 3t\left(-\frac{\cos 3t}{3}\right)$$
$$y_p = -\frac{\cos 3t}{3}\ln|\sec 3t + \tan 3t|$$
$$y = y_c + y_p$$
$$y = c_1\cos 3t + c_2 \sin 3t - \frac{\cos 3t}{3}\ln|\sec 3t + \tan 3t|$$

Problems 3

(a) Compute the Fourier series of the periodic function with a period of 2π and is given by $f(t) = t/2$ for $-\pi < t < \pi$, by evaluating the integral expressions for a_n and b_n (or using even/odd to decide that one or another of these sets of numbers are all zero).

As the function is odd, only b_n exist.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$
$$b_n = \left(\frac{1}{\pi}\right)\int_{-\pi}^{\pi}\frac{t}{2}\sin(nt)dt$$
$$b_n = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} t\sin(nt)dt\right)$$
$$b_n = \frac{1}{2\pi}\left(\left(\frac{-t}{n}\cos(nt)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos(nt)dt\right)\right)$$
$$b_n = \frac{1}{2\pi}\left(\left(\frac{-\pi}{n}\cos n\pi - \frac{-\pi}{n}\cos n\pi - \frac{1}{n^2}\sin(nt)\Big|_{-\pi}^{\pi}\right)\right)$$
$$b_n = \frac{1}{2\pi}\left(\left(\frac{-\pi}{n}\cos n\pi - \frac{-\pi}{n}\cos n\pi - 0\right)\right), \quad \because \sin n\pi = 0$$
$$b_n = \frac{1}{n}(\cos n\pi)$$
$$b_n = \frac{(-1)^{n+1}}{n}$$

Therefore, Fourier series will be

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$
$$f(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(n\pi)$$

(b) Go to the following mathlet link:
http://mathlets.org/mathlets/fourier-coefficients/
Press "Formula" to see the significance of the sliders. Move them around a bit and watch what happens. The yellow curve gives the sum, the white curve gives the sinusoidal function you are adding to the sum at the moment.

(i) With the settings on "Sine Series" and "All terms," select target A. Move the sliders around till you get the best fit you can eyeball. Record your results: $b_1 = \ldots$, $b_2 =. \ldots , \ldots .$

$$b_1 = \frac{3}{\pi}$$
$$b_2 = 0$$
$$b_3 = \frac{3}{9\pi}$$
$$b_4 = 0$$
$$b_5 = \frac{3}{25\pi}$$
$$b_6 = 0$$

(ii) Now select target D and do the same. But then, before you record your results, select "Distance." This makes a number appear above the graph, which gives a measure of the goodness of fit of the partial Fourier series you have built. Move the sliders from the top one to the bottom one to get the best fit you can. Record the results. Notice that you began with large period and then worked your way down to small period.

$$b_1 = 1$$
$$b_2 = \frac{-1}{4}$$
$$b_3 = \frac{1}{9}$$
$$b_4 = \frac{-1}{16}$$
$$b_5 = \frac{1}{25}$$
$$b_6 = \frac{-1}{36}$$
$$\text{Distance} = \; 0.42144$$

Now press "Reset," and do the same thing from the bottom up: you are putting in the best possible multiples of $\sin(6t)$, then $\sin(5t)$, and so on, in that order. Are the numbers you obtain the same as the ones you got going in the other direction? How do these values match up with what you computed in Part (a)? Do you suppose you would get different answers if you put in terms in some other more random order?
Yes, the numbers are the same. The values are similar to the ones computed in Part (a). No, the answer would be the same if we put terms in random order.

Problem 4

Consider the differential equation
$$4x^2y''(x) + 17y(x) = \frac{1}{\sqrt{x}}$$
defined for $x \geq 1$ with initial conditions $y(1) = -1$ and $y'(1) = -\frac{1}{2}$. The homogeneous solutions are given as
$$y_1(x) = \sqrt{x}\cos(2\ln x),$$
$$y_2(x) = \sqrt{x}\sin(2\ln x).$$

(a) Show that y_1 and y_2 are linearly independent.
Two functions are linearly independent if the following equation only holds for $a = b = 0$.
$$ay_1(x) + by_2(x) = 0$$
$$a\sqrt{x}\cos(2\ln x) + b\sqrt{x}\sin(2\ln x) = 0$$
For all values of x, this equation is true only when $a = b = 0$ so, y_1 and y_2 are linearly independent.

(b) Find the values of x for which Wronskian $W = 0$.

The Wronskian $W(y_1, y_2)$ associated with y_1 and y_2 is the function

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\
&= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\
&= \sqrt{x}\cos(2\ln x)\left(\frac{2\cos(2\ln x)}{\sqrt{x}} + \frac{\sin(2\ln x)}{2\sqrt{x}}\right) - \left(\frac{\cos(2\ln x)}{2\sqrt{x}} - \frac{2\sin(2\ln x)}{\sqrt{x}}\right)\sqrt{x}\sin(2\ln x) \\
&= 2\cos^2(2\ln x) + \frac{1}{2}\cos(2\ln x)\sin(2\ln x) - \frac{1}{2}\cos(2\ln x)\sin(2\ln x) + 2\sin^2(2\ln x) \\
&= 2\cos^2(2\ln x) + 2\sin^2(2\ln x) \\
&= 2\left(\cos^2(2\ln x) + \sin^2(2\ln x)\right) \\
&= 2
\end{aligned}$$

$W = 2$ no matter what x is, so there is no value of x for which $W = 0$.

(c) Using variation of parameters, find the particular solution y_p for Wronskian $W \neq 0$.

If a second order linear differential equation with variable coefficient is in its standard form

$$y'' + p(x)y' + q(x)y = f(x),$$

its particular solution is

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

where

$$u_1(x) = -\int \frac{y_2(x)f(x)}{W(y_1, y_2)}dx$$

and

$$u_2(x) = \int \frac{y_1(x)f(x)}{W(y_1, y_2)}dx$$

The given differential equation is

$$4x^2y''(x) + 17y(x) = \frac{1}{\sqrt{x}}.$$

We need to convert it into the standard form to evaluate u_1 and u_2, as follows

$$y''(x) + \frac{17}{4x^2}y(x) = \frac{1}{4x^2\sqrt{x}}.$$

For u_1, we need $y_2 = \sqrt{x}\sin(2\ln x)$, $f(x) = \dfrac{1}{4x^2\sqrt{x}}$ and $W = 2$,

$$\implies u_1 = -\int \frac{\sin(2\ln x)}{8x^2}dx$$

Let $\theta = \ln x \implies x = e^{\theta} \implies dx = e^{\theta} d\theta$

$$\implies u_1 = -\frac{1}{8}\int \frac{\sin 2\theta}{e^{2\theta}} e^{\theta} d\theta$$
$$= -\frac{1}{8}\int e^{-\theta}\sin 2\theta d\theta$$
$$= -\frac{1}{8}\,\text{Im}\left\{\int e^{(-1+2i)\theta} d\theta\right\}$$
$$= -\frac{1}{8}\,\text{Im}\left\{\frac{e^{(-1+2i)\theta}}{-1+2i}\right\}$$
$$= -\frac{1}{8}\,\text{Im}\left\{\frac{e^{(-1+2i)\theta}}{-1+2i}\left(\frac{1+2i}{1+2i}\right)\right\}$$
$$= -\frac{1}{8}\,\text{Im}\left\{\frac{e^{-\theta}(\cos 2\theta + i\sin 2\theta)(1+2i)}{-1-4}\right\}$$
$$= \frac{1}{40}e^{-\theta}(2\cos 2\theta + \sin 2\theta)$$
$$= \frac{1}{40}e^{-\ln x}(2\cos(2\ln x) + \sin(2\ln x))$$
$$\implies u_1 = \frac{1}{40x}(2\cos(2\ln x) + \sin(2\ln x))$$

Now for u_2, we need $y_1 = \sqrt{x}\cos(2\ln x)$, $f(x) = \dfrac{1}{4x^2\sqrt{x}}$ and $W = 2$,

$$\implies u_2 = -\int \frac{\cos(2\ln x)}{8x^2} dx$$
$$\implies u_2 = \frac{1}{8}\int e^{-\theta}\cos 2\theta d\theta$$
$$= \frac{1}{8}\,\text{Re}\left\{\int e^{(-1+2i)\theta} d\theta\right\}$$
$$= -\frac{1}{40}\,\text{Re}\left\{e^{-\theta}(\cos 2\theta + i\sin 2\theta)(1+2i)\right\}$$
$$= -\frac{1}{40}e^{-\theta}(\cos 2\theta - 2\sin 2\theta)$$

$$\implies u_2 = -\frac{1}{40x}\left(\cos(2\ln x) - 2\sin(2\ln x)\right)$$

The particular solution is

$$y_p = u_1 y_1 + u_2 y_2$$
$$= \frac{1}{40x}\Big((2\cos(2\ln x) + \sin(2\ln x))\sqrt{x}\cos(2\ln x) - (\cos(2\ln x) - 2\sin(2\ln x))\sqrt{x}\sin(2\ln x)\Big)$$
$$= \frac{\sqrt{x}}{40x}\left(2\cos^2(2\ln x) + \sin(2\ln x)\cos(2\ln x) - \cos(2\ln x)\sin(2\ln x) + 2\sin^2(2\ln x)\right)$$
$$= \frac{1}{40\sqrt{x}}\left(2\cos^2(2\ln x) + 2\sin^2(2\ln x)\right)$$
$$\implies y_p(x) = \frac{1}{20\sqrt{x}}$$

(d) Find the overall solution $y(x)$.

The general solution of non-homogeneous differential equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

$$y(x) = c_1 \sqrt{x} \cos(2 \ln x) + c_2 \sqrt{x} \sin(2 \ln x) + \frac{1}{20\sqrt{x}}$$

Given $y(1) = -1$,

$$y(1) = c_1 \sqrt{1} \cos(2 \ln 1) + c_2 \sqrt{1} \sin(2 \ln 1) + \frac{1}{20} = -1$$

$$\implies c_1 = -1 - \frac{1}{20} = -\frac{21}{20}$$

$$y'(x) = \frac{c_1 \cos(2 \ln x)}{2\sqrt{x}} + \frac{2c_2 \cos(2 \ln x)}{\sqrt{x}} - \frac{2c_1 \sin(2 \ln x)}{\sqrt{x}} + \frac{c_2 \sin(2 \ln x)}{2\sqrt{x}} - \frac{1}{40x\sqrt{x}}$$

Given $y'(1) = -\frac{1}{2}$,

$$y'(1) = \frac{c_1}{2} + 2c_2 - \frac{1}{40} = -\frac{1}{2}$$

$$\implies c_2 = \frac{1}{40}$$

The overall solution becomes

$$y(x) = -\frac{\sqrt{x}}{40} \left(42 \cos(2 \ln x) - \sin(2 \ln x) + \frac{2}{x} \right)$$

Problems 5

Find a solution to the following differential equations:

(a) $x^2y'' + 10xy' + 6y = \ln x^2$

$$x^2y'' + 10xy' + 6y = \ln x^2$$

With substitution $x = e^t$ or $t = \ln x$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Substituting back the values

$$x^2\left(\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right) + 10x\left(\frac{1}{x}\frac{dy}{dt}\right) + 6y = \ln(e^t)^2$$

$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 6y = 2t$$

Now, we will first solve the associated homogeneous differential equation to find y_c

$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 6y = 0$$

$$r^2 + 9r + 6 = 0$$

$$r = \frac{-9 - \sqrt{57}}{2},\ \frac{-9 + \sqrt{57}}{2}$$

$$y_c = c_1 e^{\frac{-9-\sqrt{57}}{2}t} + c_2 e^{\frac{-9+\sqrt{57}}{2}t}$$

Let $y_p = At + B$

$$y_p' = A$$

$$y_p' = 0$$

Putting back the values in equation

$$0 + 9A + 6(At + B) = 2t$$

Comparing coefficients

$$6A = 2$$

$$A = \frac{1}{3}$$

$$9A + 6B = 0$$

$$B = -\frac{1}{2}$$

$$y_p = \frac{t}{3} - \frac{1}{2}$$

$$y = y_c + y_p$$

$$y = c_1 e^{\frac{-9-\sqrt{57}}{2}t} + c_2 e^{\frac{-9+\sqrt{57}}{2}t} + \frac{t}{3} - \frac{1}{2}$$

Putting back the substitution $t = \ln x$

$$y = c_1 e^{\frac{-9-\sqrt{57}}{2}\ln x} + c_2 e^{\frac{-9+\sqrt{57}}{2}\ln x} + \frac{\ln x}{3} - \frac{1}{2}$$

(b) $x^2y'' - 3xy' + 13y = 4 + 3x$

$$x^2y'' - 3xy' + 13y = 4 + 3x$$

With substitution $x = e^t$ or $t = \ln x$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Substituting back the values

$$x^2\left(\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right) - 3x\left(\frac{1}{x}\frac{dy}{dt}\right) + 13y = 4 + 3e^t$$

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 4 + 3e^t$$

Now, we will first solve the associated homogeneous differential equation to find y_c

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 0$$

$$r^2 - 4r + 13 = 0$$

$$r = 2 + 3\iota,\ 2 - 3\iota$$

$$y_c = e^{2t}(c_1\cos 3t + c_2\sin 3t)$$

Considering only $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 4$

Let $y_{p1} = A$

$$y'_{p1} = 0$$

$$y''_{p1} = 0$$

Putting back the values in equation

$$0 + 0 + 13(A) = 4$$

Comparing coefficients

$$13A = 4$$

$$A = \frac{4}{13}$$

$$y_{p1} = \frac{4}{13}$$

Considering only $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 3e^t$

Let $y_{p2} = Ae^t$

$$y'_{p2} = Ae^t$$

$$y''_{p2} = Ae^t$$

Putting back the values in equation

$$0Ae^t - 4Ae^t + 13Ae^t = 3e^t$$

$$10Ae^t = 3e^t$$

Comparing coefficients

$$10A = 3$$

$$A = \frac{3}{10}$$

$$y_{p2} = \frac{3}{10}e^t$$

From superposition principle

$$y_p = y_{p1} + y_{p2}$$

$$y_p = \frac{4}{13} + \frac{3}{10}e^t$$

$$y = y_c + y_p$$

$$y = e^{2t}(c_1 \cos 3t + c_2 \sin 3t) + \frac{4}{13} + \frac{3}{10}e^t$$

Putting back the substitution $t = \ln x$

$$y = e^{2(\ln x)}(c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)) + \frac{4}{13} + \frac{3}{10}e^{\ln x}$$

$$y = x^2(c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)) + \frac{4}{13} + \frac{3}{10}x$$