MT232: Differential Equations

Homework 4 Solutions

Due Mon Nov 26, 9:00 AM *Fall 2018*

Problem 1

For each of the following initial and boundary value problems,

(a) $y'' + 4y' + 5y = 0, \quad y(1) = 3$ and $y'(1) = 9$

(i) Determine whether system is stable, marginally stable or unstable.
The char. Eqn. is given as

$$S(r) = r^2 + 4r + 5$$

Roots of the above char eqn are

$$r_1, r_2 = -2 \pm i$$

Since α = -2<0, $\implies$ System is stable.

(ii) For stable and marginally stable systems, find value of natural frequency (ω_o), damping ratio (ζ) and quality factor (Q) .
As we know the general equation

$$y'' + 2\zeta\omega_o y' + \omega_o^2 y = 0$$

By comparing the coefficients. we get

$$\omega_o = \sqrt{5} = 2.24$$

$$2\zeta\omega_o = 4 \implies \zeta = \frac{4}{2\sqrt{5}} = 0.894$$

To find quality factor, the genral equation is given as

$$y'' + \frac{\omega_o}{Q}y' + \omega_o^2$$

$$\frac{\omega_o}{Q} = 4 \implies Q = \frac{\sqrt{5}}{4} = 0.56$$

(iii) Is the system undamped, underdamped, overdamped or critically damped?
Since $0 < \zeta = 0.894 < 1$, $\implies$ the system is underdamped.

(iv) In case of underdamped system, find the pseudo-natural frequency of the system.
In case of underdamped system, the damped frequency ω_d is called pseudo-frequency of the system.

$$\omega_d = \omega_o\sqrt{1 - \zeta^2} = 1$$

(v) Solve the differential equation for the given initial or boundary conditions. In case of sinusoidal solutions, write the solution in $A\cos(wt - \phi)$ form.

$$y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t)$$

$$y(1) = e^{-2}(c_1 \cos(1) + c_2 \sin(1))$$

$$0.0731c_1 + 0.114c_2 = 3 \quad (1)$$

$$y'(t) = -e^{-2t}c_1(2\cos t + \sin t) - e^{-2t}c_2(2\sin t - \cos t)$$

Putting $y'(1) = 9$, we get

$$0.2601c_1 - 0.155c_2 = 9 \quad (2)$$

Solving Eq(1) and Eq(2), we get

$$c_1 = -81.43$$

$$c_2 = 78.53$$

Finally we get,

$$y(t) = e^{-2t}(-81.43\cos(t) + 78.53\sin(t))$$

$$y(t) = 113.13e^{-2t}cos(t - 2.374)$$

(vi) Sketch the graph of the solution.

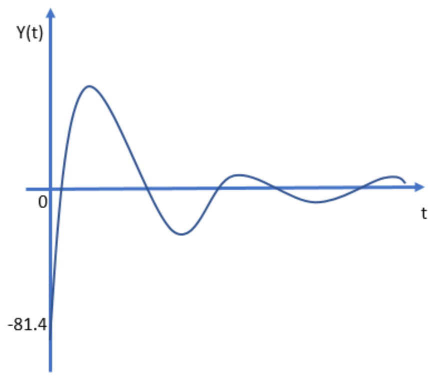

Figure 1: Underdamped Response.

(b) $y'' - 4y' + 4y = 0, \quad y(1) = 1$ and $y'(1) = 1$

(i) Determine whether system is stable, marginally stable or unstable.

The char. Eqn. is given as

$$S(r) = r^2 - 4r + 4$$

Roots of the above char eqn are

$$r_1 = r_2 = r = 2$$

Since r = 2>0, $\implies$ System is unstable.

(ii) Solve the differential equation for the given initial or boundary conditions. In case of sinusoidal solutions, write the solution in $A\cos(wt - \phi)$ form.

$$y(t) = (c_1 + c_2t)e^{2t}$$

$$y(1) = (c_1 + c_2)e^2$$

$$(c_1 + c_2)e^2 = 1 \quad (3)$$

$$y'(t) = 2e^{2t}c_1 + e^{2t}(2t + 1)c_2$$

Putting $y'(1) = 1$, we get

$$2e^2c_1 + 3e^2c_2 = 1 \quad (4)$$

Solving Eq(3) and Eq(4), we get

$$c_1 = 0.271$$

$$c_2 = -0.1353$$

Finally we get,

$$y(t) = (0.271 - 0.1353t)e^{2t}$$

(iii) Sketch the graph of the solution.

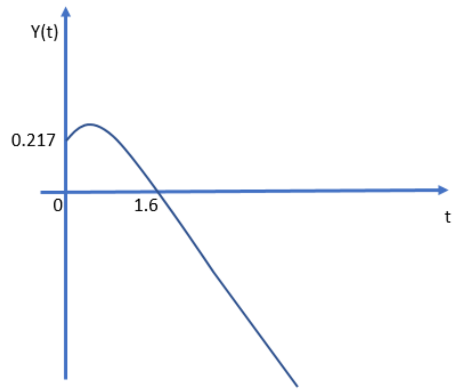

Figure 2: Unstable System Response.

(c) $y'' + 5y' + 6y = 0, \quad y(0) = 1$ and $y'(0) = 0$

(i) Determine whether system is stable, marginally stable or unstable.
The char. Eqn. is given as

$$S(r) = r^2 + 5r + 6$$

Roots of the above char eqn are

$$r_1 = -2$$

$$r_2 = -3$$

Since $r_1 = -2 < 0$ and $r_2 = -3 < 0 \implies$ System is stable.

(ii) For stable and marginally stable systems, find value of natural frequency (ω_o), damping ratio (ζ) and quality factor (Q) .
As we know the general equation

$$y'' + 2\zeta\omega_o y' + \omega_o^2 y = 0$$

By comparing the coefficients. we get

$$\omega_o = \sqrt{6} = 2.45$$

$$2\zeta\omega_o = 5 \implies \zeta = \frac{5}{2\sqrt{6}} = 1.021$$

To find quality factor, the genral equation is given as

$$y'' + \frac{\omega_o}{Q} y' + \omega_o^2$$

$$\frac{\omega_o}{Q} = 5 \implies Q = \frac{\sqrt{6}}{5} = 0.489$$

(iii) Is the system undamped, underdamped, overdamped or critically damped?
Since $\zeta = 1.021 > 1, \implies$ the system is Overdamped.

(iv) Solve the differential equation for the given initial or boundary conditions. In case of sinusoidal solutions, write the solution in $A\cos(wt - \phi)$ form.

$$y(t) = c_1e^{-2t} + c_2e^{-3t}$$

$$y(0) = c_1 + c_2$$

$$c_1 + c_2 = 1 \tag{5}$$

$$y'(t) = -2c_1e^{-2t} - 3c_2e^{-3t}$$

Putting $y'(0) = 0$, we get

$$-2c_1 - 3c_2 = 0 \tag{6}$$

Solving Eq(5) and Eq(6), we get

$$c_1 = 3$$

$$c_2 = -2$$

Finally we get,

$$y(t) = 3e^{-2t} - 2e^{-3t}$$

(v) Sketch the graph of the solution.

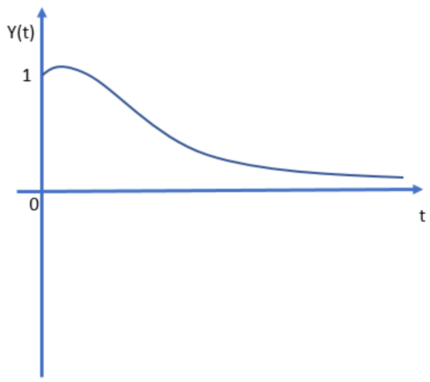

Figure 3: Overdamped Response.

(d) $4y'' + y' - 4y = 0, \quad y(0) = 0$ and $y'(0) = 1$

(i) Determine whether system is stable, marginally stable or unstable.
The char. Eqn. is given as

$$S(r) = 4r^2 + r - 4$$

Roots of the above char eqn are

$$r_1 = 0.883$$

$$r_2 = -1.133$$

Since $r_1 = 0.883 > 0 \implies$ System is unstable.

(ii) Solve the differential equation for the given initial or boundary conditions. In case of sinusoidal solutions, write the solution in $A\cos(wt - \phi)$ form.

$$y(t) = c_1e^{0.883t} + c_2e^{-1.133t}$$

$$y(0) = c_1 + c_2$$

$$c_1 + c_2 = 0 \tag{7}$$

$$y'(t) = 0.883c_1e^{0.883t} - 1.133c_2e^{-1.133t}$$

Putting $y'(0) = 1$, we get

$$0.883c_1 - 1.133c_2 = 1 \tag{8}$$

Solving Eq(7) and Eq(8), we get

$$c_1 = 0.496$$

$$c_2 = -0.496$$

Finally we get,

$$y(t) = 0.496e^{0.883t} - 0.496e^{-1.133t}$$

(iii) Sketch the graph of the solution.

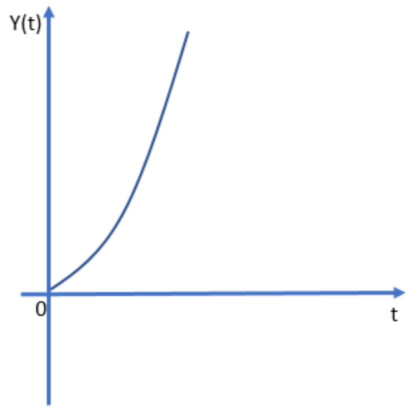

Figure 4: Unstable System Response.

(e) $y'' + y' - 6y = 0, \quad y(0) = 10$ and $y'(0) = 0$

(i) Determine whether system is stable, marginally stable or unstable.
The char. Eqn. is given as

$$S(r) = r^2 - r - 6$$

Roots of the above char eqn are

$$r_1 = 3$$

$$r_2 = -2$$

Since $r_1 = 3 > 0 \implies$ System is unstable.

(ii) Solve the differential equation for the given initial or boundary conditions. In case of sinusoidal solutions, write the solution in $A\cos(wt - \phi)$ form.

$$y(t) = c_1e^{3t} + c_2e^{-2t}$$

$$y(0) = c_1 + c_2$$

$$c_1 + c_2 = 10 \tag{9}$$

$$y'(t) = 3c1e^{3t} - 2e^{-2t}$$

Putting $y'(0) = 0$, we get

$$3c_1 - 2c_2 = 0 \tag{10}$$

Solving Eq(9) and Eq(10), we get

$$c_1 = 4$$

$$c_2 = 6$$

Finally we get,

$$y(t) = 4e^{3t} + 6e^{-2t}$$

(iii) Sketch the graph of the solution.

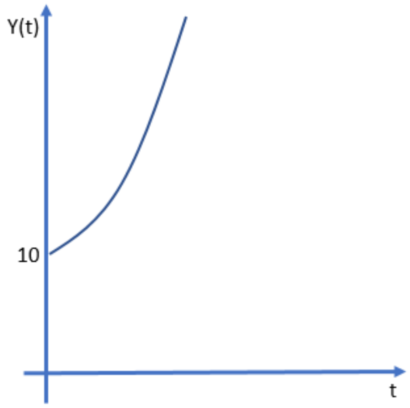

Figure 5: Unstable System Response.

Problem 2

Figure 6 shows a cart of mass m attached to a fixed spring with stiffness k. The cart is shown in its equilibrium position but it can have a displacement x towards right or left causing a resulting extension or compression in the spring. Given that $m = 3$ kg and $k = 12$ Nm^{-1}, use these values for all parts of this problem unless stated otherwise.

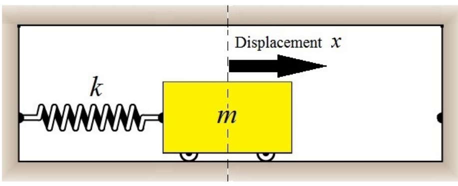

Figure 6: Ideal spring mass system with no friction or damping

(a) Write down a linear second order differential equation for the system, in terms of m, k, and x.

The differential equation of the above system can be written as

$$m\frac{d^2x}{dt^2} + kx = 0$$

(b) The cart is pulled 0.5 m to the right and released at $t = 0$. It is expected to oscillate about its equilibrium position with a natural frequency ω_0.

(i) Find $x(t)$ for $t \geq 0$ by solving the differential equation. What is the natural frequency ω_0?

Converting the differential equation to its standard form

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{11}$$

The characteristic equation is

$$S(r) = r^2 + \frac{k}{m} = 0$$

$$\implies r = \pm i\sqrt{\frac{k}{m}} = \pm i\sqrt{\frac{12}{3}} = \pm 2i$$

Because the equation is homogeneous, the general solution can be written as

$$x(t) = c_1 \cos 2t + c_2 \sin 2t$$

The cart is pulled 0.5 m to the right and released from rest at $t = 0$, which means that the initial conditions are $x(0) = 0.5,\ x'(0) = 0$

$$c_1 \cos 0 + c_2 \sin 0 = 0.5 \implies c_1 = 0.5$$

$$x(t) = 0.5 \cos 2t + c_2 \sin 2t$$

$$x'(t) = -\sin 2t + 2c_2 \cos t$$

$$x'(0) = -\sin 0 + 2c_2 \cos 0 = 0 \implies c_2 = 0$$

$$x(t) = \frac{1}{2} \cos 2t.$$

Natural frequency $\omega_0 = 2$, i.e. the coefficient of t in $\cos 2t$.

(ii) Determine damping and stability condition of system.
Damping Condition: Since the damping ratio $\zeta = 0 \implies$ the system is undamped i.e No Damping.
Stability Condition: Since $\alpha = 0 \implies$ the system is maginally stable.

(iii) Sketch the graph of $x(t)$, clearly showing the amplitude and time period of the oscillation.

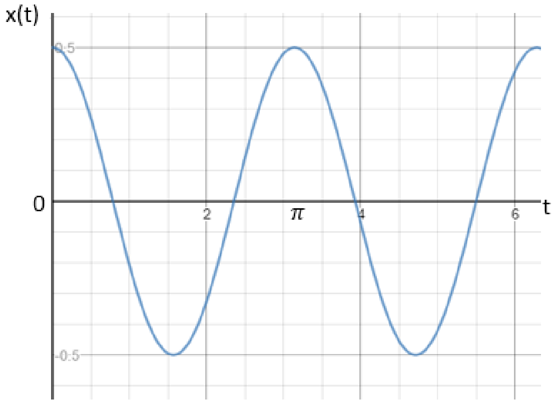

Figure 7: Undamped Response.

Suppose that now a dashpot with damping coefficient b is attached to the cart as shown in Figure 8. We have three dashpots of three different damping coefficients available. We attach the dashpots one by one and examine the effect of each dashpot on the motion of the cart.

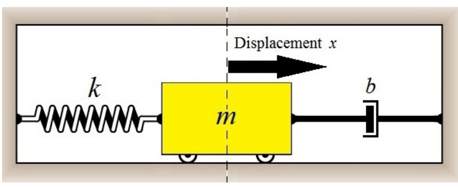

Figure 8: Spring mass system connected to a dashpot attached for damping

(a) Write down a linear second order differential equation for this system, in terms of m, b, k and x.
Since there is a dashpot attached, there will be drag force (damping force) $F_d = -b\frac{dx}{dt}$.Incorporating this damping term, the differential equations becomes

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

(b) First, the dashpot with $b = 6$ kgs^{-1} is attached to the cart.

(a) Determine whether system is stable, marginally stable or unstable.
Converting the differential equation to its standard form

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

The characteristic equation is

$$P(r) = r^2 + \frac{b}{m}r + \frac{k}{m} = 0$$

$$\implies r = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} = -\frac{b}{2m} \pm \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

For $k = 12,\ m = 3,\ b = 6$

$$r = -1 \pm i\sqrt{3} = \alpha \pm i\omega_d$$

Since $\alpha = -1 < 0 \implies$ system is stable.

(b) For stable and marginally stable system, find value of (ω_o) and damping ratio (ζ).
The DE in std. form is

$$x'' + 2x' + 4 = 0$$

the general form of DE is

$$y'' + 2\zeta\omega_o y' + \omega_o^2 y = 0$$

By camparing the coefficients, we get

$$\omega_o = 2$$

$$2\zeta\omega_o = 2 \implies \zeta = 0.5$$

(c) What is the damping condition of system (undamped, underdamped, overdamped or critically damped).
Since $0 < \zeta = 0.5 < 1 \implies$ the system is underdamped.

(d) The cart is once again pulled 0.5 m to the right and released at $t = 0$. Find $x(t)$ for $t \geq 0$ by solving the differential equation and sketch the graph of $x(t)$.

Since $r = -1 \pm i\sqrt{3}$. the general solution is

$$x(t) = e^{-t}\left(c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t\right)$$

Using initial conditions $x(0) = 0.5$ and $x'(0) = 0$,

$$x(0) = c_1 \cos 0 + c_2 \sin 0 = 0.5 \implies c_1 = 0.5$$

$$x'(0) = -c_1 + \sqrt{3}c_2 = 0 \implies c_2 = \frac{1}{2\sqrt{3}}$$

$$x(t) = \frac{1}{2}e^{-t}\left(\cos\sqrt{3}t + \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right)$$

It can be converted into the form

$$x(t) = \frac{1}{2}e^{-t}\sqrt{\left(1^2 + \frac{1}{3}\right)}\cos\left(\sqrt{3}t - \tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$x(t) = \frac{1}{\sqrt{3}}e^{-t}\cos\left(\sqrt{3}t - \frac{\pi}{6}\right)$$

Damped natural frequency $\omega_d = \sqrt{3}$ rad s^{-1}.

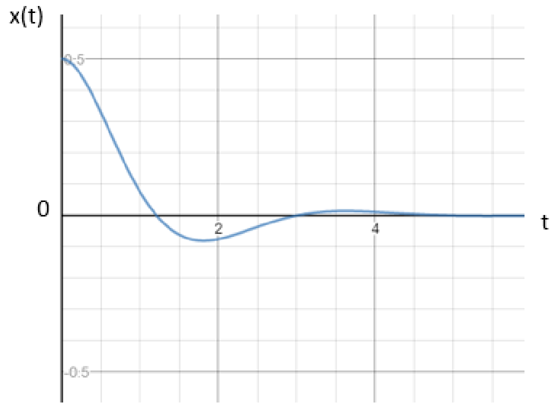

Figure 9: Underdamped Response.

(c) Repeat part (b) for the second dashpot with $b = 12$ kgs^{-1}.

(a) Determine whether system is stable, marginally stable or unstable.
For $k = 12,\ m = 3,\ b = 12$, here we get repeated roots as follow;

$$r_1 = r_2 = r = -2$$

Since $r = -2 < 0 \implies$ system is stable.

(b) For stable and marginally stable system, find value of (ω_o) and damping ratio (ζ).
The DE in std. form is

$$x'' + 4x' + 4 = 0$$

the general form of DE is

$$y'' + 2\zeta\omega_o y' + \omega_o^2 y = 0$$

By camparing the coefficients, we get

$$\omega_o = 2$$

$$2\zeta\omega_o = 4 \implies \zeta = 1$$

(c) What is the damping condition of system (undamped, underdamped, overdamped or critically damped).
Since $\zeta = 1 \implies$ the system is Critically damped.

(d) The cart is once again pulled 0.5 m to the right and released at $t = 0$. Find $x(t)$ for $t \geq 0$ by solving the differential equation and sketch the graph of $x(t)$.

Since $r = -2$. the general solution is

$$x(t) = (c_1 + c_2 t)e^{-2t}$$

Using initial conditions $x(0) = 0.5$ and $x'(0) = 0$,

$$x(0) = 0.5 \implies c_1 = 0.5$$

$$x'(0) = 0 \implies c_2 = 1$$

$$x(t) = \frac{1}{2}e^{-2t}(1 + 2t)$$

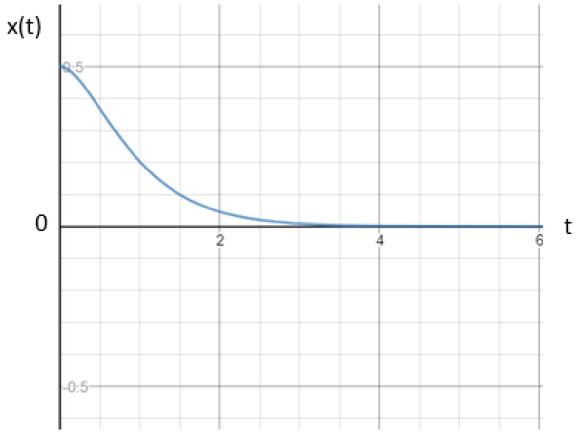

Figure 10: Critically damped Response.

(d) Repeat part (b) for the third dashpot with $b = 24$ kgs^{-1}.

(a) Determine whether system is stable, marginally stable or unstable.
For $k = 12,\ m = 3,\ b = 24$, here we get following roots;
$r = -4 \pm 2\sqrt{3}$ Since $r_1 < 0$ and $r_2 < 0 \implies$ system is stable.

(b) For stable and marginally stable system, find value of (ω_o) and damping ratio (ζ).
The DE in std. form is

$$x'' + 8x' + 4 = 0$$

the general form of DE is

$$y'' + 2\zeta\omega_o y' + \omega_o^2 y = 0$$

By camparing the coefficients, we get

$$\omega_o = 2$$

$$2\zeta\omega_o = 8 \implies \zeta = 2$$

(c) What is the damping condition of system (undamped, underdamped, overdamped or critically damped).
Since $\zeta = 2 > 1 \implies$ the system is Overdamped.

(d) The cart is once again pulled 0.5 m to the right and released at $t = 0$. Find $x(t)$ for $t \geq 0$ by solving the differential equation and sketch the graph of $x(t)$.

Since $r = -4 \pm 2\sqrt{3}$. the general solution is

$$x(t) = c_1 e^{\left(-4-2\sqrt{3}\right)t} + c_2 e^{\left(-4+2\sqrt{3}\right)t}$$

Using initial conditions $x(0) = 0.5$ and $x'(0) = 0$,

$$x(0) = 0.5 \implies c_1 + c_2 = 0.5$$

$$x'(0) = 0 \implies \left(-4 - 2\sqrt{3}\right) c_1 + \left(-4 + 2\sqrt{3}\right) c_2 = 0$$

$$\implies c_1 = -0.0387, \quad c_2 = 0.5387$$

$$x(t) = e^{-4t}(-0.0387e^{-2\sqrt{3}t} + 0.5387e^{2\sqrt{3}t})$$

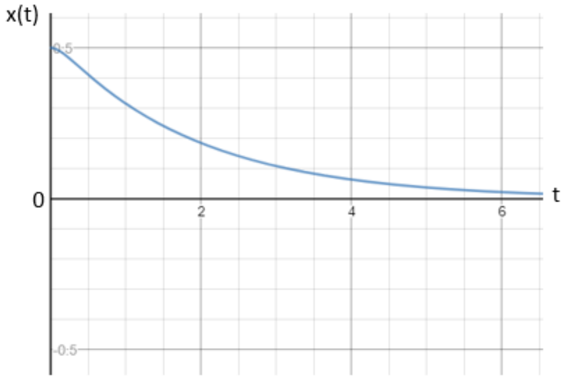

Figure 11: Overdamped Response.

(e) Comment on the response of $x(t)$ find in part (b,c and d). Explain your answer.
In part b, when b = 6, the system was underdamped, from the solution curve we can see that initially there are some oscillations in the response that decay with time.
In part c, when b =12, the system was critically damped, from the solution curve we can see that there are no oscillations in the response and the system settles to its mean position very fast.
In part d, when b = 24, the system was Overdamped, from the solution curve we can see that there are no oscillations in the response and the system takes some time to settles to its mean position.

(f) Which $x(t)$ will settle in minimum time?
For $b = 12$, the oscillations are critically damped hence the cart settles fastest to its equilibrium position.

Problem 3

In this problem you will understand the concept of tuning a circuit to a frequency of the received signal, which can be the frequency of a radio station or the frequency at which your mobile service provider transmits.

(a) LC circuit

(b) RLC circuit

Figure 12

(a) Figure 12(a) shows an LC circuit driven by a variable voltage source $V(t)$, which causes a current $I(t)$ to flow through the circuit.

(i) From your notes, write down the differential equation for current I in the circuit, in terms of L, C and V. Convert the equation into its standard form.
Using Kirchhoff's Voltage Law, we can write the associated differential equation as

$$LI''(t) + \frac{1}{C}I(t) = V'(t)$$

$$\implies I''(t) + \frac{1}{LC}I(t) = \frac{V'(t)}{L} \tag{12}$$

(ii) Write down the characteristic equation $S(r)$ for the differential equation and find its roots in terms of L, C and i.
Characteristic equation associated with homogeneous equation of Eq. (12) is

$$S(r) = r^2 + \frac{1}{LC} = 0$$

$$\implies r = \pm i\frac{1}{\sqrt{LC}}$$

(iii) Using your answer to (ii), find the natural frequency ω_0 of the circuit?
The complementary solution is going to be

$$y_c(t) = c_1 \cos\left(\frac{1}{\sqrt{LC}}\, t\right) + c_2 \sin\left(\frac{1}{\sqrt{LC}}\, t\right)$$

Hence, $\omega_0 = \dfrac{1}{\sqrt{LC}}$ is the natural frequency.

(iv) Rewrite the differential equation in terms of ω_0, instead of L and C.

Hence the equation will come

$$I''(t) + \omega_0^2 I(t) = \frac{V'(t)}{L}$$

(v) Let $V(t) = \cos\omega t$, where ω is the driving frequency. At what frequency ω does the resonance happen, i.e. the amplitude of current $I(t)$ grows to a maximum value?

The LC circuit is being driven by a sinusoidal electromotive force (voltage) of frequency ω. If this frequency is equal to the natural frequency ω_0 of the LC circuit, then resonance will occur. i.e $\omega = \omega_o = \frac{1}{\sqrt{LC}}$

(vi) Given $L = 1$ H and $C = 0.125$ F, calculate the value of the resonant frequency.

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(1)(0.125)}} = 2\sqrt{2} \text{ rad s}^{-1}$$

(vii) Find the expression for particular solution of current $I(t)$ in the circuit at resonant frequency, and sketch its graph.

$$I''(t) + \omega_0^2 I(t) = \frac{V'(t)}{L}$$

Because $\omega_0 = 2\sqrt{2}$, $V(t) = \cos\omega t$, $V'(t) = -\omega\sin\omega t$ and $L = 1$,

$$I''(t) + 8I(t) = -\omega\sin\omega t = -\omega\mathrm{Im}\{e^{i\omega t}\}$$

Complex particular solution

$$\tilde{I}_p = \frac{-\omega e^{i\omega t}}{P(i\omega)}$$

But at resonant frequency, $\omega = \omega_0$ and hence $P(i\omega_0) = 0$. So, the complex particular solution would instead be

$$\tilde{I}_p = \frac{-\omega_0 t e^{i\omega_0 t}}{P'(i\omega_0)} = \frac{-2\sqrt{2}te^{i2\sqrt{2}t}}{4\sqrt{2}i} = \frac{1}{2}ite^{i2\sqrt{2}t} = \frac{1}{2}it(\cos 2\sqrt{2}t + i\sin 2\sqrt{2}t)$$

The particular solution of can written as

$$I_p(t) = \mathrm{Im}\left\{\tilde{I}_p\right\} = -\frac{1}{2}t\sin 2\sqrt{2}t$$

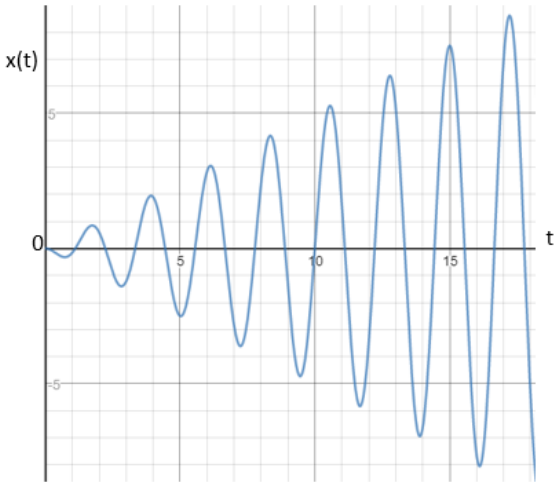

Figure 13: Response at Resonance

(b) Figure 12(b) shows an RLC circuit driven by a variable voltage source $V(t)$, which causes a current $I(t)$ to flow through the circuit. By varying the values of R, L or C, this circuit can be tuned to the frequency of a signal received as voltage $V(t)$.

(i) From your notes, write down the differential equation for current I in the circuit, in terms of R, L, C and V. Convert the equation into its standard form.

Now there is resistance in circuit. So including the effect of R, we can write the associated differential equation as

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = V'(t)$$

$$I(s) = \frac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}V(s)$$

$$\implies I''(t) + \frac{R}{L}I'(t) + \frac{1}{LC}I(t) = \frac{V'(t)}{L} \tag{13}$$

(ii) Using your notes, rewrite this equation in terms of quality factor Q and natural frequency ω_0.

$$I''(t) + \frac{\omega_0}{Q}I'(t) + \omega_0^2 I(t) = \frac{V'(t)}{L} \tag{14}$$

where $\frac{\omega_0}{Q} = \frac{R}{L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

(iii) Write down the characteristic equation $S(r)$ for the differential equation and find its roots in terms of Q, ω_0 and i.

$$S(r) = r^2 + \frac{\omega_0}{Q}r + \omega_0^2 = 0$$

$$\implies r = -\frac{\omega_0}{2Q} \pm i\omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

(iv) What is the damped natural frequency ω_d of the circuit, in terms of Q and ω_0?

$$\omega_d = \omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

(v) For $V(t) = \cos\omega t$, $R = 2\ \Omega$, $L = 1$ H and $C = 0.125$ F, evaluate ω_d. Use these values of R, L and C for the remaining parts of this problem.

$$\frac{\omega_0}{Q} = \frac{R}{L} \implies Q = \omega_0\frac{L}{R} = \sqrt{2}$$

$$\implies \omega_d = \omega_0\sqrt{1 - \frac{1}{4Q^2}} = \sqrt{7}\ \text{rad s}^{-1}$$

(vi) Find the steady-state current $I(t)$ in the circuit. (Recall that the steady-state solution is the solution as $t \to \infty$. For stable systems, steady-state solution is equal to the particular solution because the complementary solution approaches 0 as $t \to \infty$.)

Using the given values for R, L and C, differential equation can be written as

$$I''(t) + 2I'(t) + 8I(t) = -\omega\sin\omega t$$

The characteristic equation will be

$$S(r) = r^2 + 2r + 8$$

and $f(t)$ is defined as

$$f(t) = -\omega \sin \omega t = \text{Im}\left\{-\omega e^{i\omega t}\right\}$$

Hence the particular solution $I_p(t)$ will be

$$\begin{aligned}
I_p(t) &= Im\left\{\frac{-\omega e^{i\omega t}}{S(i\omega)}\right\} \\
&= Im\left\{\frac{-\omega e^{i\omega t}}{8-\omega^2+2i\omega}\left(\frac{8-\omega^2-2i\omega}{8-\omega^2-2i\omega}\right)\right\} \\
&= Im\left\{\frac{-\omega(\cos\omega t + i\sin\omega t)(8-\omega^2-2i\omega)}{\omega^4-12\omega^2+64}\right\} \\
&= \frac{\omega}{\omega^4-12\omega^2+64}\left(2\omega\cos\omega t - (8-\omega^2)\sin\omega t\right)
\end{aligned}$$

(vii) For this circuit, its resonant frequency ω_r is not exactly equal to the damped natural frequency ω_d. Resonance happens at a slightly different frequency. In this part, you will calculate ω_r. Convert $I(t)$ into the form $I(t) = A(\omega)\cos(\omega t - \phi(\omega))$, where $A(\omega)$ and $\phi(\omega)$ are functions of ω. Resonance will happen when the value of $A(\omega)$ is maximized. Find the value of ω for which A is maximized. This is the resonant frequency ω_r for the circuit. (You do not need to evaluate ϕ)

$$I_p(t) = \frac{\omega}{\omega^4-12\omega^2+64}\left(2\omega\cos\omega t - (8-\omega^2)\sin\omega t\right) \equiv A\cos(\omega t - \phi)$$

$$\implies A = \frac{\omega}{\omega^4-12\omega^2+64}\sqrt{(2\omega)^2+(8-\omega^2)^2} = \frac{\omega}{\omega^4-12\omega^2+64}\sqrt{\omega^4-12\omega^2+64}$$

$$\implies A(\omega) = \frac{\omega}{\sqrt{\omega^4-12\omega^2+64}} = \frac{\omega}{\omega\sqrt{\omega^2-12+\dfrac{64}{\omega^2}}} = \frac{1}{\sqrt{\omega^2-12+\dfrac{64}{\omega^2}}}$$

Let $B(\omega) = \omega^2 - 12 + \dfrac{64}{\omega^2}$. Maximizing $A(\omega)$ is the same as minimizing $B(\omega)$. To find the stationary points of $B(\omega)$,

$$\frac{dB}{d\omega} = 2\omega - \frac{128}{\omega^3} = 0$$

$$\omega^4 = 64 \implies \omega = 2\sqrt{2}$$

Because $\dfrac{d^2B}{d\omega^2} = 2 + \dfrac{384}{\omega^4}$ is always positive, the stationary point is a minimum.

So, the frequency at which $B(\omega)$ is minimized and hence $A(\omega)$ is maximized is the resonant frequency $\omega_r = 2\sqrt{2}$ rad s^{-1}.

(viii) Suppose the radio signal received as $V(t)$ is at a frequency $f = 2$ Hz (recall that $\omega = 2\pi f$). The RLC circuit is said to be tuned to the received radio signal when the amplitude of current $I(t)$ through the circuit is maximum. This happens when the resonant frequency f_r of the circuit equals 2 Hz, the frequency of the radio signal. The resonant frequency of the circuit can be changed by varying the values of either R, L or C. Suppose that we can only control the value of C. Keeping $R = 2\ \Omega$, $L = 1$ H, calculate the value of C for which the resonant frequency of the circuit $f_r = 2$ Hz and the circuit will be tuned to the received radio signal.

In class, we discussed that for damped systems the resonance frequency ω_r is given as $\omega_r = \omega_o\sqrt{1-2\zeta^2}$, but in this problem we are getting $\omega_r = \omega_o$. Actually in class we solved problems invloving $\sin\omega t$ or $\cos\omega t$ as forcing function but here we have ω multiplied by $\sin\omega t$ i.e $\omega\sin\omega t$ here ω_r is such that it is shifted to ω_o. If we look at the value of ω_r, we find that $\omega_r = \omega_0 = 2\sqrt{2}$.

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1C}} = 2\sqrt{2}$$

$$\implies C = 0.125 \text{ F}$$
