MT232: Differential Equations

Homework 3 Solution

Due: Fri, Oct 26, 9:00 AM *Fall 2018*

Problem 1

Find the derivative df of each of the following functions.

(a) $f(x) = x^2 + \cos\sqrt{x}$

$$\begin{aligned} \frac{df}{dx} &= 2x - \sin\sqrt{x}\left(\frac{d}{dx}\sqrt{x}\right) \\ &= 2x - \frac{\sin\sqrt{x}}{2\sqrt{x}} \\ \implies df &= \left(2x - \frac{\sin\sqrt{x}}{2\sqrt{x}}\right) dx \end{aligned}$$

(b) $f(x,y) = x^3 + 3x^2y + 3xy^2 + y^3$

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= \frac{\partial}{\partial x}\left(x^3 + 3x^2y + 3xy^2 + y^3\right) dx + \frac{\partial}{\partial y}\left(x^3 + 3x^2y + 3xy^2 + y^3\right) dy \\ \implies df &= (3x^2 + 6xy + 3y^2)dx + (3x^2 + 6xy + 3y^2)dy \end{aligned}$$

(c) $f(x,y,z) = x\cos(yz) - \sin(xz) - z\ln(xy)$

$$\begin{aligned} df &= \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \\ &= \frac{\partial}{\partial x}(x\cos(yz) - \sin(xz) - z\ln(xy))dx \\ &\quad + \frac{\partial}{\partial y}(x\cos(yz) - \sin(xz) - z\ln(xy))dy \\ &\quad + \frac{\partial}{\partial z}(x\cos(yz) - \sin(xz) - z\ln(xy))dz \\ &= \left(\cos(yz) - \cos(xz)\frac{\partial}{\partial x}(xz) - \left(\frac{z}{xy}\right)\frac{\partial}{\partial x}(xy)\right) dx \\ &\quad + \left(-x\sin(yz)\frac{\partial}{\partial y}(yz) - \left(\frac{z}{xy}\right)\frac{\partial}{\partial y}(xy)\right) dy \\ &\quad + \left(-x\sin(yz)\frac{\partial}{\partial z}(yz) - \cos(xz)\frac{\partial}{\partial z}(xz) - \ln(xy)\right) dz \\ \implies df &= \left(\cos(yz) - z\cos(xz) - \frac{z}{x}\right) dx \\ &\quad + \left(-xz\sin(yz) - \frac{z}{y}\right) dy \\ &\quad + (-xy\sin(yz) - x\cos(xz) - \ln(xy))\, dz \end{aligned}$$

Problem 2

Solve the following initial value problems.

(a) $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$, $y(1) = 2$

$$x\frac{dy}{dx} + y = y^{-2}$$

This given equation is a Bernoulli equation which has the following form:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

$$\text{Substitution: } z = y^{1-n} = y^3$$
$$\frac{dz}{dx} = 3y^2\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{dz}{dx}.\frac{1}{3y^2}$$

Putting the value of $\dfrac{dy}{dx}$

$$\frac{x}{3y^2}\frac{dz}{dx} + y = y^{-2}$$
$$\frac{dz}{dx} + \frac{3y^2}{x}.y = \frac{3y^2}{x}.y^{-2}$$
$$\frac{dz}{dx} + \frac{3y^3}{x} = \frac{3}{x}$$
$$\frac{dz}{dx} + \frac{3}{x}z = \frac{3}{x} \quad (z = y^3)$$
$$\text{Integrating Factor } = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

Multiplying both sides by the integrating factor,

$$x^3\frac{dz}{dx} + x^3\frac{3}{x}z = x^3\frac{3}{x}$$
$$(x^3z)' = 3x^2$$
$$x^3z = \int 3x^2dx$$
$$x^3z = x^3 + c$$
$$z = 1 + \frac{c}{x^3} = \frac{x^3 + c}{x^3}$$

Putting back the value of z,

$$y^3 = \frac{x^3 + c}{x^3}$$

Putting initial conditions: $y(1) = 2$,

$$8 = \frac{1 + c}{1} \implies c = 7$$
$$y^3 = \frac{x^3 + 7}{x^3}$$
$$\implies y = \sqrt[3]{\frac{x^3 + 7}{x^3}}$$

(b) $(y^2 \cos x - 3x^2 y - 2x)dx + (2y \sin x - x^3 + \ln y)dy = 0, \; y(0) = e$

$$\text{In this question, } M(x,y) = (y^2 \cos x - 3x^2 y - 2x)$$
$$N(x,y) = (2y \sin x - x^3 + \ln y)$$

Now we will check the condition for the exact equation

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y^2 \cos x - 3x^2 y - 2x)$$
$$= 2y \cos x - 3x^2$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2y \sin x - x^3 + \ln y)$$
$$= 2y \cos x - 3x^2$$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$$f(x,y) = \int M(x,y)dx = \int (y^2 \cos x - 3x^2 y - 2x)dx$$
$$f(x,y) = y^2 \sin x - x^3 y - x^2 + g(y)$$
$$\frac{\partial f}{\partial y} = 2y \sin x - x^3 + g'(y)$$

Comparing with N we get:

$$g'(y) = \ln y$$
$$g(y) = \int \ln y$$
$$g(y) = y \ln y - y$$
$$\text{So, } f(x,y) = y^2 \sin x - x^3 y - x^2 + y \ln y - y$$
$$c = y^2 \sin x - x^3 y - x^2 + y \ln y - y$$

Putting initial conditions: $y(0) = e$

$$c = 0$$

$$\implies y^2 \sin x - x^3 y - x^2 + y \ln y - y = 0$$

(c) $\frac{dy}{dx} = \frac{(x+y)^2}{1-2xy-x^2}$, $y(1) = 1$

$$(x+y)^2 dx + (x^2 + 2xy - 1)dy = 0$$

In this question, $M(x,y) = (x+y)^2$

$$N(x,y) = (x^2 + 2xy - 1)$$

Now we will check the condition for the exact equation

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x+y)^2 = 2(x+y) = 2x + 2y$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2xy - 1) = 2x + 2y$$

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

$$f(x,y) = \int N(x,y)dy = \int (x^2 + 2xy - 1)dy$$

$$f(x,y) = x^2y + xy^2 - y + g(x)$$

$$\frac{\partial f}{\partial x} = 2xy + y^2 + g'(x)$$

Comparing with $M(x,y) = (x+y)^2 = x^2 + y^2 + 2xy$

$$g'(x) = x^2$$

$$g(x) = \int x^2 dx$$

$$g(x) = \frac{x^3}{3}$$

So, $f(x,y) = x^2y + xy^2 - y + \frac{x^3}{3}$

$$c = x^2y + xy^2 - y + \frac{x^3}{3}$$

Putting initial conditions $y(1) = 1$

$$c = 1 + 1 - 1 + \frac{1}{3}$$

$$= \frac{4}{3}$$

$$\implies x^2y + xy^2 - y + \frac{x^3}{3} = \frac{4}{3}$$

(d) $x\dfrac{dy}{dx} + y = y^2x^2 \ln x,\ y(1) = -1$

This given equation is a Bernoulli equation which has the following form:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

$$\text{Substitution: } z = y^{1-2} = y^{-1}$$

$$\frac{dz}{dx} = -\frac{1}{y^2}.\frac{dy}{dx}$$

$$\frac{dy}{dx} = -y^2\frac{dz}{dx}$$

$$\text{Putting the value of } \frac{dy}{dx}$$

$$-xy^2\frac{dz}{dx} + y = y^2x^2 \ln x$$

$$\frac{dz}{dx} - \frac{1}{xy^2}y = -\frac{1}{xy^2}y^2x^2 \ln x$$

$$\frac{dz}{dx} - \frac{1}{xy} = -x \ln x$$

$$\frac{dz}{dx} - \frac{z}{x} = -x \ln x \qquad \left(z = \frac{1}{y}\right)$$

$$\text{Integrating factor: } e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$$

Multiplying both sides by the integrating factor,

$$\frac{1}{x}.\frac{dz}{dx} - \frac{1}{x}.\frac{z}{x} = -\frac{1}{x}.x \ln x$$

$$\left(z.\frac{1}{x}\right)' = -\ln x$$

$$z.\frac{1}{x} = \int -\ln x dx$$

$$z.\frac{1}{x} = -x \ln x + x + c$$

$$z = -x^2 \ln x + x^2 + cx$$

$$\text{Putting back the value of } z,\ \frac{1}{y} = -x^2 \ln x + x^2 + cx$$

$$y = \frac{1}{-x^2 \ln x + x^2 + cx}$$

$$\text{Putting initial conditions: } y(1) = -1$$

$$-1 = \frac{1}{1+c} \implies c = -2$$

$$\implies y = \frac{1}{-x^2 \ln x + x^2 - 2x}$$

$$y = \frac{1}{x(x - x \ln x - 2)}$$

Problem 3

The differential equation $\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$ is known as Riccati equation has a variety of applications in physics and engineering such as supersymmetric quantum mechanics, variational calculus, nonlinear physics, optimal control systems and thermodynamics. The Riccati equation can be solved by two successive substitution given we know a solution. If y_1 is a solution, the first substitution $y = y_1 + u$ reduces Riccati equation to a Bernoulli equation. Find one parameter family of solutions for the following differential equation:

$$\frac{dy}{dx} = -\frac{4}{x^2} - \frac{1}{x}y + y^2$$

where $y_1 = \frac{2}{x}$ is a known solution of the equation.

First substitution: $y = y_1 + u$

$$y = \frac{2}{x} + u \implies u = y - \frac{2}{x}$$

$$\frac{du}{dx} = \frac{dy}{dx} + \frac{2}{x^2} \implies \frac{dy}{dx} = \frac{du}{dx} - \frac{2}{x^2}$$

Putting the value of $\frac{dy}{dx}$ and y :

$$\frac{du}{dx} - \frac{2}{x^2} = -\frac{4}{x^2} - \frac{1}{x}\left(\frac{2}{x} + u\right) + \left(\frac{2}{x} + u\right)^2$$

$$\frac{du}{dx} - \frac{2}{x^2} = -\frac{4}{x^2} - \frac{2}{x^2} - \frac{u}{x} + \frac{4}{x^2} + u^2 + \frac{4u}{x}$$

$$\frac{du}{dx} = u^2 + \frac{3u}{x}$$

$$\implies \frac{du}{dx} - \frac{3u}{x} = u^2$$

Second substitution: $z = u^{1-n} = u^{-1}$

$$\frac{dz}{dx} = -\frac{1}{u^2}\frac{du}{dx}$$

$$\frac{du}{dx} = -u^2\frac{dz}{dx}$$

Putting the value of $\frac{du}{dx}$

$$-u^2\frac{dz}{dx} - \frac{3u}{x} = u^2$$

$$\frac{dz}{dx} + \frac{3u}{xu^2} = -1$$

$$\frac{dz}{dx} + \frac{3z}{x} = -1 \quad (z = u^{-1})$$

Integrating factor $= e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

$$(x^3 z)' = -x^3$$

$$x^3 z = \int -x^3 dx = -\frac{x^4}{4} + c$$

$$z = -\frac{x}{4} + \frac{c}{x^3}$$

Putting back the value of z,

$$\frac{1}{u} = -\frac{x}{4} + \frac{c}{x^3}$$
$$u = \left(-\frac{x}{4} + \frac{c}{x^3}\right)^{-1}$$

Putting back the value of u,

$$y - \frac{2}{x} = \left(-\frac{x}{4} + \frac{c}{x^3}\right)^{-1}$$

$$\implies y = \frac{2}{x} + \left(-\frac{x}{4} + \frac{c}{x^3}\right)^{-1}$$

Problem 4

Consider the following initial value problem.

$y' = ty^2 - \frac{y}{t},\ y(1) = 1$

(a) Solve the differential equation to find its closed-form solution $y(t)$.

$$y' + \frac{y}{t} = ty^2$$

This given equation is a Bernoulli equation which has the following form:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

$$z = y^{1-n} = y^{-1}$$
$$\frac{dz}{dt} = -\frac{1}{y^2}\frac{dy}{dt} \implies \frac{dy}{dt} = -y^2\frac{dz}{dt}$$
$$-y^2\frac{dz}{dt} + \frac{y}{t} = ty^2$$
$$\frac{dz}{dt} - \frac{1}{yt} = -t$$
$$\frac{dz}{dt} - \frac{z}{t} = -t \quad (z = y^{-1})$$

Integrating Factor: $e^{\int -\frac{1}{t}} = e^{-\ln t} = e^{\ln t^{-1}} = t^{-1}$

$$\left(\frac{1}{t}z\right)' = -1$$
$$\frac{1}{t}z = -\int dt = -t + c$$
$$\implies z = -t^2 + ct$$

Putting the value of z,

$$\frac{1}{y} = -t^2 + ct$$
$$\implies y = \frac{1}{-t^2 + ct}$$

Putting the initial conditions: $y(1) = 1$

$$1 = \frac{1}{-1 + c} \implies c = 2$$

$$\implies y = \frac{1}{2t - t^2} = \frac{1}{t(2-t)}$$

(b) Use Euler's method to obtain a four decimal-place approximation of $y(1.2)$ using

(i) a step size of 0.1

$$\begin{aligned} t_{n+1} &= t_n + h \\ y_{n+1} &= y_n + hf(t_n, y_n) \\ &= y_n + h\left(t_n y_n^2 - \frac{y_n}{t_n}\right) \end{aligned}$$

Step 0:

$$t_0 = 1, \quad y_0 = 1$$

Step 1:

$$\begin{aligned} t_1 &= 1.1 \\ y_1 &= y_0 + hf(t_0, y_0) \\ &= 1 + 0.1(0) \\ &= 1 \end{aligned}$$

Step 2:

$$\begin{aligned} t_2 &= 1.2 \\ y_2 &= y_1 + hf(t_1, y_1) \\ &= 1 + 0.1(0.1909) \\ &= 1.0191 \end{aligned}$$

(ii) a step size of 0.05

$$\begin{aligned} t_{n+1} &= t_n + h \\ y_{n+1} &= y_n + hf(t_n, y_n) \end{aligned}$$

Step 0:

$$t_0 = 1, \quad y_0 = 1$$

Step 1:

$$\begin{aligned} t_1 &= 1.05 \\ y_1 &= y_0 + hf(t_0, y_0) \\ &= 1 + 0.05(0) \\ &= 1 \end{aligned}$$

Step 2:

$$\begin{aligned} t_2 &= 1.1 \\ y_2 &= y_1 + hf(t_1, y_1) \\ &= 1 + 0.05(0.0976) \\ &= 1.0049 \end{aligned}$$

Step 3:

$$\begin{aligned} t_3 &= 1.15 \\ y_3 &= y_2 + hf(t_2, y_2) \\ &= 1.0049 + 0.05(0.1972) \\ &= 1.0147 \end{aligned}$$

Step 4:

$$\begin{aligned} t_4 &= 1.2 \\ y_4 &= y_3 + hf(t_3, y_3) \\ &= 1.1972 + 0.05(0.3018) \\ &= 1.0298 \end{aligned}$$

(c) Now use the midpoint (RK2) method to obtain a four decimal-place approximation of $y(1.2)$ using

(i) a step size of 0.1

$$\begin{aligned} t_{n+1} &= t_n + h \\ y_{n+1} &= y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{hf(t_n, y_n)}{2}\right) \end{aligned}$$

Step 0:

$$t_0 = 1, \quad y_0 = 1$$

Step 1:

$$\begin{aligned}
t_1 &= 1.1 \\
y_1 &= y_0 + hf\left(t_0 + \frac{h}{2}, y_0 + \frac{hf(t_0, y_0)}{2}\right) \\
&= 1 + 0.1f\left(1.05, 1 + \frac{0.1(0)}{2}\right) \\
&= 1 + 0.1(0.0976) \\
&= 1.0098
\end{aligned}$$

Step 2:

$$\begin{aligned}
t_2 &= 1.2 \\
y_2 &= y_1 + hf\left(t_1 + \frac{h}{2}, y_1 + \frac{hf(t_1, y_1)}{2}\right) \\
&= 1.0098 + 0.1f\left(1.15, 1.0098 + \frac{0.1(0.2036)}{2}\right) \\
&= 1.0098 + 0.1(0.3094) \\
&= 1.0407
\end{aligned}$$

(ii) a step size of 0.05

$$\begin{aligned}
t_{n+1} &= t_n + h \\
y_{n+1} &= y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{hf(t_n, y_n)}{2}\right)
\end{aligned}$$

Step 0:

$$t_0 = 1, \quad y_0 = 1$$

Step 1:

$$\begin{aligned}
t_1 &= 1.05 \\
y_1 &= y_0 + hf\left(t_0 + \frac{h}{2}, y_0 + \frac{hf(t_0, y_0)}{2}\right) \\
&= 1 + 0.05f\left(1.0025, 1 + \frac{0.05(0)}{2}\right) \\
&= 1 + 0.05(0.0494) \\
&= 1.0025
\end{aligned}$$

Step 2:

$$\begin{aligned}
t_2 &= 1.1 \\
y_2 &= y_1 + hf\left(t_1 + \frac{h}{2}, y_1 + \frac{hf(t_1, y_1)}{2}\right) \\
&= 1.0025 + 0.05f\left(1.0525, 1 + \frac{0.05(0.1005)}{2}\right) \\
&= 1.0025 + 0.05(0.1509) \\
&= 1.0100
\end{aligned}$$

Step 3:

$$\begin{aligned}
t_3 &= 1.15 \\
y_3 &= y_2 + hf\left(t_2 + \frac{h}{2}, y_2 + \frac{hf(t_2, y_2)}{2}\right) \\
&= 1.0100 + 0.05f\left(1.1025, 1 + \frac{0.05(0.2039)}{2}\right) \\
&= 1.0100 + 0.05(0.2569) \\
&= 1.0229
\end{aligned}$$

Step 4:

$$\begin{aligned}
t_4 &= 1.2 \\
y_4 &= y_3 + hf\left(t_3 + \frac{h}{2}, y_3 + \frac{hf(t_3, y_3)}{2}\right) \\
&= 1.0229 + 0.05f\left(1.1525, 1 + \frac{0.05(0.3137)}{2}\right) \\
&= 1.0229 + 0.05(0.3711) \\
&= 1.0414
\end{aligned}$$

(d) Now use the RK4 method (`ode45()` function in MATLAB) to obtain a four decimal-place approximation of $y(1.2)$ using

(i) a step size of 0.1

The following lines of codes in MATLAB produce the desired result

```
tspan = [1 1.1 1.2];
y0 = 1;
[t,y] = ode45(@(t,y) t*y^2-y/t, tspan, y0);
```

You can get the following values by typing `y`

$y_0 = 1$

$y_1 = 1.0101$
$y_2 = 1.0417$

(ii) a step size of 0.05

The following lines of codes in MATLAB produce the desired result

```
tspan = [1 1.05 1.1 1.15 1.2];
y0 = 1;
[t,y] = ode45(@(t,y) t*y^2-y/t, tspan, y0);
```

You can get the following values by typing `y`

$y_0 = 1$
$y_1 = 1.0025$
$y_2 = 1.0101$
$y_3 = 1.0230$
$y_2 = 1.0417$

Scientifically, the accuracy of your estimated values can be determined by a metric known as mean squared error defined as

$$\text{MSE} = \frac{1}{n+1}\sum_{i=0}^{n}(y(t_i) - y_i)^2$$

where $y(t_i)$ is an actual value of y at time t_i, and y_i is the estimated value at the i^{th} iteration step.

(e) Evaluate the MSEs for $y(1), y(1.1)$ and $y(1.2)$ in

(i) b(i).

(ii) c(i).

(iii) d(i).

Step#	t_i	Analytic $y(t_i)$	Euler's y_i	RK2's y_i	RK4's y_i	Euler's $(y(t_i) - y_i)^2$	RK2's $(y(t_i) - y_i)^2$	RK4's $(y(t_i) - y_i)^2$
0	1	1	1	1	1	0	0	0
1	1.1	1.0101	1	1.0098	1.0101	0.0001	0.0000	0.0000
2	1.2	1.0417	1.0191	1.0407	1.0417	0.0005	0.0000	0.0000

(i) $\text{MSE} = \frac{0 + 0.0001 + 0.0005}{3} = \frac{0.0006}{3} = 0.0002$

(ii) $\text{MSE} = \frac{0 + 0.0000 + 0.0000}{3} = \frac{0.0000}{3} = 0.0000$

(iii) $\text{MSE} = \frac{0 + 0.0000 + 0.0000}{3} = \frac{0.0000}{3} = 0.0000$

(f) Evaluate the MSEs for $y(1), y(1.05), y(1.1), y(1.15)$ and $y(1.2)$ in

(i) b(ii).

(ii) c(ii).

Step#	t_i	Analytic $y(t_i)$	Euler's y_i	RK2's y_i	RK4's y_i	Euler's $(y(t_i) - y_i)^2$	RK2's $(y(t_i) - y_i)^2$	RK4's $(y(t_i) - y_i)^2$
0	1	1	1	1	1	0	0	0
1	1.05	1.0025	1	1.0025	1.0025	0.0000	0.0000	0.0000
2	1.1	1.0101	1.0049	1.0100	1.0101	0.0000	0.0000	0.0000
3	1.15	1.0230	1.0147	1.0229	1.0230	0.0001	0.0000	0.0000
4	1.2	1.0417	1.0298	1.0414	1.0417	0.0001	0.0000	0.0000

(iii) d(ii).

(i) $\text{MSE} = \dfrac{0 + 0.0000 + 0.0000 + 0.0001 + 0.0001}{5} = \dfrac{0.0002}{5} = 0.0000$

(ii) $\text{MSE} = \dfrac{0 + 0.0000 + 0.0000 + 0.0000 + 0.0000}{5} = \dfrac{0.0000}{5} = 0.0001$

(iii) $\text{MSE} = \dfrac{0 + 0.0000 + 0.0000 + 0.0000 + 0.0000}{5} = \dfrac{0.0000}{5} = 0.0000$

(g) Using the insights from parts (e) and (f)

(i) Explain how decreasing the step size affects the accuracy of the estimated solution?
The value of MSE decreased from 0.0109 to 0.0056 and from 0.0004 to 0.0001 in Euler's and RK2 method respectively when we decreased the step size. Hence, decreasing the step size also decrease MSE.

(ii) Rank the three methods based on their accuracy. Give appropriate reasoning based on the values of MSE.
The raking of the methods is:

i. RK4
ii. RK2
iii. Euler

The lower the MSE, the better the approximation. The value of MSE is minimum for RK4 followed by the MSE of RK2. The MSE of Euler's Method is the highest.
