MT232: Differential Equations

Homework 2 Solution

Due Fri Oct 12, 9:00 AM *Fall 2018*

Problem 1

When certain kinds of chemicals are combined, the reactions are autonomous and the rate at which the new compound is formed is modeled by the differential equation

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X)$$

where $k > 0$ is a constant of proportionality and $\beta > \alpha > 0$. Here $X(t)$ denotes the number of grams of the new compound formed in time t

(a) Use a phase portrait of the differential equation to predict the behavior of $X(t)$ as $t \to \infty$

For equilibrium points

$$\frac{dX}{dt} = 0$$

$$k(\alpha - X)(\beta - X) = 0 \implies X = \alpha, X = \beta$$

Now for stability

$$0 < X < \alpha \implies \frac{dX}{dt} = k(+)(+) > 0$$

$$\alpha < X < \beta \implies \frac{dX}{dt} = k(-)(+) < 0$$

$$X > \beta \implies \frac{dX}{dt} = k(-)(-) > 0$$

therefore α is stable and β is unstable.

(b) Consider the case when $\alpha = \beta$. Use a phase portrait of the differential equation to predict the behavior of X(t) as $t \to \infty$ when $X(0) = \alpha > 0$.

Here $\alpha = \beta$, the system becomes

$$\frac{dX}{dt} = k(\alpha - X)^2$$

the equilibrium point is only α

$$0 < X < \alpha \implies \frac{dX}{dt} = k(+)(+) > 0$$

$$X > \alpha \implies \frac{dX}{dt} = k(+)(+) > 0$$

hence the equilibrium point α becomes semi stable

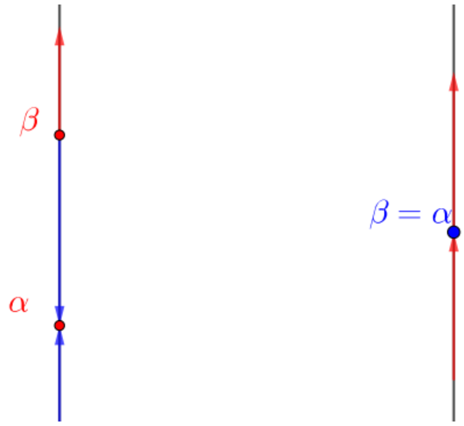
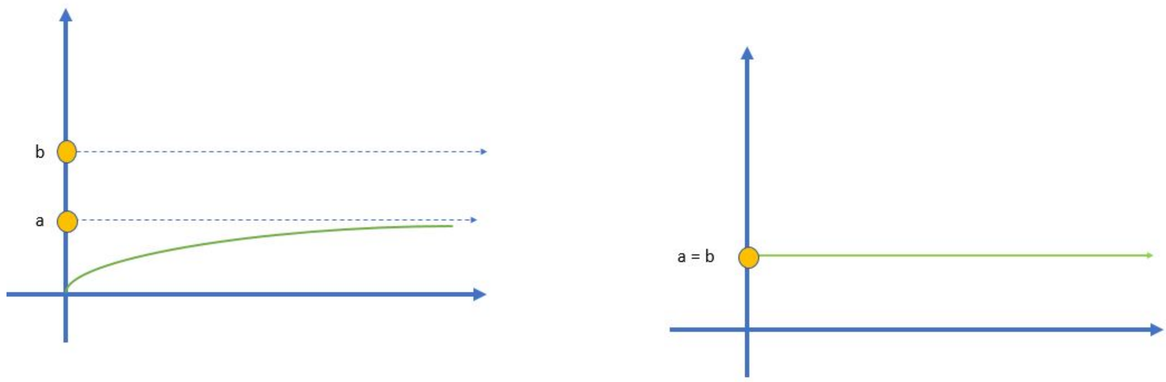


Figure 1: Phase portrait



(a) (b)

Figure 2: Solution Curves for (a) and (b)

(c) Find the explicit solution of the DE in the case when $k = 1$ and $\alpha = \beta = 5$ for

(i) $X(0) = \dfrac{\alpha}{2}$

(ii) $X(0) = 2\alpha$.

Graph these two solutions.

This DE can be solved by Separation of Variables, when k = 1 and $\alpha = \beta = 5$. The general solotion of DE is given by

$$\frac{dX}{dt} = (5 - X)^2$$

$$\frac{dX}{(5 - X)^2} = dt$$

Integrating on both sides, we get;

$$\frac{1}{5 - X} = t + c$$

$$5 - X = \frac{1}{t + c}$$

$$X(t) = 5 - \frac{1}{t + c}$$

for (i) $X(0) = \alpha/2$, $c = 0.4$, so the solution becomes

$$X(t) = 5 - \frac{1}{t + 0.4}$$

for (ii) $X(0) = 2\alpha$, $c =$ -0.2, so the solution becomes

$$X(t) = 5 - \frac{1}{t - 0.2}$$

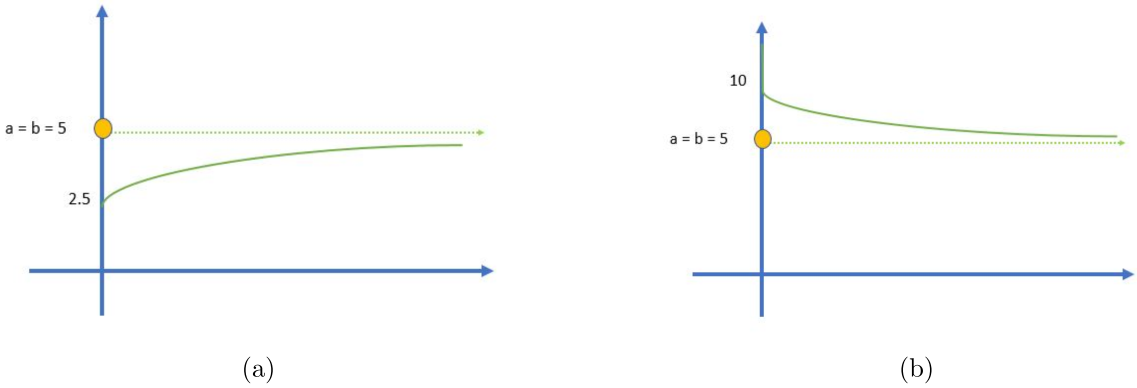


Figure 3: Graphs of above cases

(d) Does the behavior of the solutions as $t \to \infty$ agree with your answers to part (b)?
Yes, from the above two graphs it is clearly shown that the behavior of the solution agrees with answers in part (b). Mathematically it can be illustrated as below;

$$\lim_{t\to\infty} X(t) = 5 - \frac{1}{t - 0.2}$$

$$X(t) = 5$$

Problem 2

Consider a body is falling from the top of the tower of Pisa under the influence of gravity and air resistance. The mass of the body is m, v is its speed, g is the acceleration due to gravity and b is the air drag coefficient.

(a) Derive the DE of the above model. (Note: Take downward direction positive)
The DE of the above model is given by

$$ma = F_{net}$$

Taking downward direction positive

$$ma = mg - bv$$

$$ma = mg - bv$$

$$m\frac{dv}{dt} = mg - bv$$

(b) Show that the differential equation represents an autonomous system.
The DE in standard form is given by

$$\frac{dv}{dt} = g - \frac{b}{m}v$$

Since the right hand side of the above DE does not explicitly depend on independent variable t, hence the DE is autonomous.

(c) Find and classify the equilibrium point of the system and draw its phase portrait.

At equilibrium points, $\frac{dv}{dt} = 0$. So the required equilibrium point is

$$g - \frac{b}{m}v = 0$$

$$v = \frac{mg}{b}$$

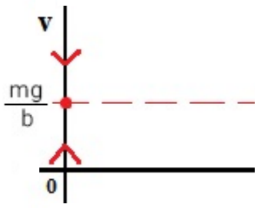


Figure 4: Phase portrait for 2(b)

For $v > \frac{mg}{b}$, let's assume $v = \frac{2mg}{b}$

$$\frac{dv}{dt} = g - \frac{b}{m}\left(\frac{2mg}{b}\right) = -g < 0$$

Hence the phase arrow will point downwards.

For $v < \frac{mg}{b}$, let's assume $v = 0$

$$\frac{dv}{dt} = g - \frac{b}{m}(0) = g > 0$$

Hence the phase arrow will point upwards. Both arrows are pointing towards the equilibrium point, hence the equilibrium point is stable.

(d) Now consider that from the top of the tower of Pisa, a ping-pong ball of mass 5 g is being dropped from rest. What happens to the speed of the ball during the fall, considering the coefficient of air drag $b = 0.47$?

At equilibrium point,

$$v = \frac{mg}{b} = \frac{(0.005)(9.81)}{0.47} = 0.10426 \text{ ms}^{-1}$$

The speed of ball increases from rest during the fall until it approaches terminal speed or hits the ground. But because the terminal speed is very small and Tower of Pisa is tall enough, it most definitely approached the terminal speed before hitting the ground.

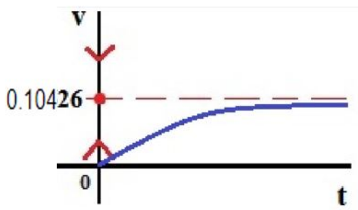


Figure 5: Phase portrait for 2(d)

(e) In another trial, the ball is shot down at an initial speed of 5 ms^{-1}. Now during the fall, what happens to the speed of the ball?
The equilibrium point will stay the same. Because $5 > 0.10426$, the speed of the ball decreases during the fall until it approaches terminal speed or hits the ground. But because the terminal speed is close to initial speed 5 and Tower of Pisa is tall enough, it most definitely approached the terminal speed before hitting the ground.

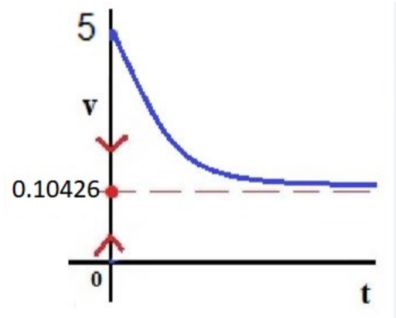


Figure 6: Phase portrait for 2(e)

(f) In yet another trial, a bouncy ball is dropped from rest which is of the same shape and size but 10 times heavier than the ping-pong ball. Does this ball reach the ground faster than the ping-pong ball? Explain.
Terminal velocity for this 50 g ball

$$v = \frac{(0.05)(9.81)}{0.47} = 1.0426\text{ms}^{-1}.$$

Yes, this ball reaches the ground much faster because of its higher terminal velocity which will be achieved soon after dropping the ball.

(g) But if we drop two balls of masses 20 kg and 200 kg from the tower at once, why do they reach the ground almost at the same time, although one is 10 times heavier than the other?
Terminal velocity for 10 kg ball would be

$$v = \frac{(10)(9.81)}{0.47} = 208.7\text{ms}^{-1},$$

and terminal velocity for 100 kg ball would be

$$v = \frac{(100)(9.81)}{0.47} = 2087\text{ms}^{-1},$$

in both cases terminal velocoties are very high. During a drop from such a height, both of these balls will be far from reaching there respective terminal velocities before hitting the ground. Hence they will fall over the same ranges of speed and hence hit the ground almost at the same time.

Problem 3

Consider the following differential equation with initial condition $(1, 2)$.

$$x\frac{dy}{dx} + y = 2y^2$$

Solve the differential equation using an appropriate method you have studied so far until Oct 5.
Solution: Separation of Variables

$$\frac{dy}{dx} = \frac{y(2y-1)}{x}$$

$$\frac{dy}{y(2y-1)} = \frac{dx}{x}$$

The left hand side of the above equation is converted in to partial fractions as follow;

$$\frac{1}{y(2y-1)} = -\frac{1}{y} + \frac{2}{(2y-1)}$$

$$(-\frac{1}{y} + \frac{2}{(2y-1)})dy = \frac{1}{x}dx$$

Integrating on both sides, we get

$$-\ln y + ln(2y-1) = \ln x + c1$$

After rearranging,

$$\frac{2y-1}{y} = cx$$

Hence the general solution is

$$y = \frac{1}{2-cx}$$

After putting the initial condition (1,2), we get

$$y = \frac{2}{4-3x}$$

Problem 4

Figure 7(a) shows a circuit in which an inductor of inductance L is connected in series with a resistor R. Let's assume that the voltage across inductor and the current flowing through the inductor at time t are $v(t)$ and $i(t)$ respectively. Given that $R = 2\Omega$ and $L = 0.2$H.

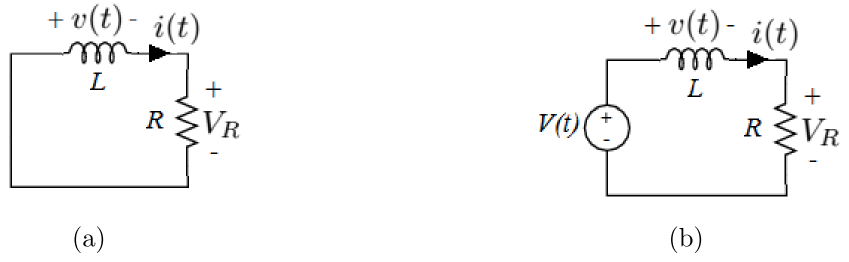


Figure 7: RL circuits for Problem 4

(a) Using Kirchhoff's Voltage Law, write down the differential equation of voltage $v(t)$ in the circuit given in Figure 7(a).

$$v(t) + V_R(t) = 0$$

$$v(t) + i(t)R = 0$$

Since $i(t) = \frac{1}{L}\int v(t)dt$

$$v(t) + \frac{R}{L}\int v(t)dt = 0$$

$$\frac{dv(t)}{dt} + \frac{R}{L}v(t) = 0$$

(b) At $t = 0$, a current of 5 A was passing through the inductor. Without solving the differential equation, sketch V_R for $t = 0$ to $t \to \infty$. Mark clearly on your graph, the values of V_R at $t = 0$ and $t \to \infty$.

The plot can be drawn using a phase portrait because

$$\frac{dv(t)}{dt} = -v(t)\frac{R}{L}$$

represents an autonomous system. To find the equilibrium points

$$\frac{dv(t)}{dt} = -v(t)\frac{R}{L} = 0$$

$$\implies v(t) = 0$$

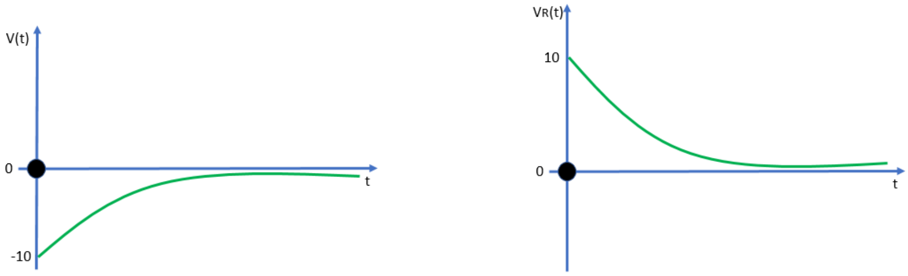
is a single equilibrium point. For $v > 0$, $\frac{dv}{dt} < 0$ and for $v < 0$, $\frac{dv}{dt} > 0$. Hence this equilibrium point is stable. For initial condition $i(0) = 5$,

$$v(t) + i(t)R = 0$$

$$v(0) + i(0)R = 0$$

$$v(0) = -i(0)R = -10$$

Since $V_R(t) = -v(t)$, Plot of $v(t)$ and $V_R(t)$ will be as follow;



(a) (b)

Figure 8: Plots of $v(t)$ and $V_R(t)$

(c) For the conditions in (a), guess the exact solution of the differential equation, evaluating any constants involved.

$$\frac{dv(t)}{dt} = -v(t)\frac{R}{L} = -10v(t)$$

$$\implies v(t) = ce^{-10t}v(0) = c = -10v(t) = -10e^{-10t}$$

is the solution of the differential equation. Voltage across the resistor

$$V_R(t) = -v(t) = 10e^{-10t}.$$

Now a voltage source $V(t)$ is added in series to the circuit as shown in Figure 7(b).

(d) Write down the differential equation of voltage $v(t)$ in the circuit given in Figure 7(b).

$$v(t) + V_R(t) = V(t)$$

Similarly we can make DE as follows;

$$\frac{dv(t)}{dt} + \frac{R}{L}v(t) = \frac{V(t)}{L}$$

(e) The voltage source is set to give a constant DC voltage of $V(t) = 5$ V. Given that initially a current of 1 A is passing through the circuit. Without solving the differential equation, sketch V_R for $t = 0$ to $t \to \infty$. Mark clearly on your graph, the values of V_R at $t = 0$ and $t \to \infty$.
The plot can be drawn using a phase portrait because

$$\frac{dv(t)}{dt} = \frac{V(t)}{L} - v(t)\frac{R}{L} = 25 - 10v(t)$$

represents an autonomous system. To find the equilibrium points

$$\frac{dv(t)}{dt} = 25 - 10v(t) = 0$$

$$\implies v(t) = 2.5$$

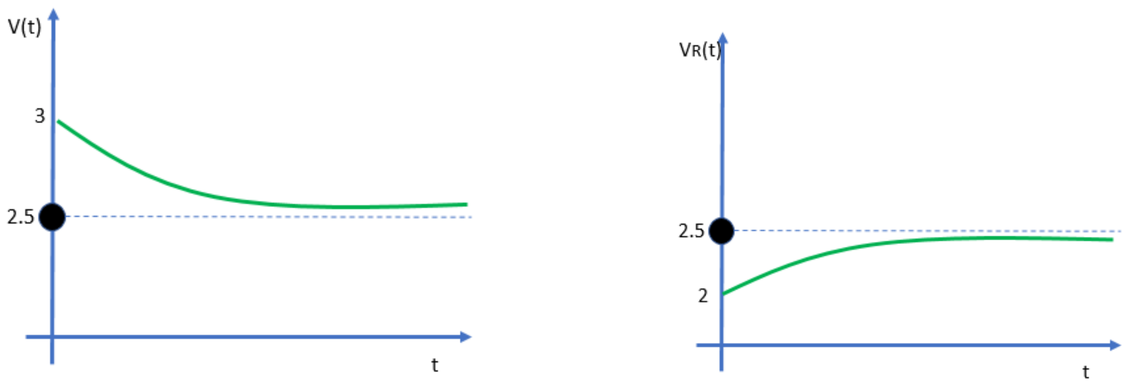
is a single equilibrium point. For $v > 2.5$, $\frac{dv}{dt} < 0$ and for $v < 2.5$, $\frac{dv}{dt} > 0$. Hence this equilibrium point is stable. For initial condition $i(0) = 1$,

$$v(0) + i(0)R = V(0)$$

$$v(0) + (1)(2) = 5$$

$$v(0) = 5 - 2 = 3$$

Since $V_R(t) = V(t) - v(t) = 5$ - $v(t)$. Plots of $v(t)$ and $V_R(t)$ will be as follow;



(a) (b)

Figure 9: Plots of $v(t)$ and $V_R(t)$

(f) Express this linear differential equation in the standard form discussed in the lecture. For $V(t) = 5$ V, solve the differential equation using an integrating factor and find the voltage signal $V_R(t)$, evaluating any constants involved. Verify that your solution agrees with your plot in (c).

$$\frac{dv(t)}{dt} = 25 - 10v(t)$$

$$\frac{dv}{dt} + 10v = 25$$

$$\mu\frac{dv}{dt} + 10\mu v = 25\mu$$

Integrating factor $\mu = e^{\int 10dt} = e^{10t}$.

$$e^{10t}\frac{dv}{dt} + 10e^{10t}v = 25e^{10t}$$

$$\frac{d}{dt}(e^{10t}v) = 25e^{10t}$$

$$e^{10t}v = 25\int e^{10t}dt + c$$

$$e^{10t}v(t) = 25\frac{e^{10t}}{10} + c$$

$$v(t) = 2.5 + ce^{-10t}.$$

Using the initial condition v(0) = 3,

$$v(0) = 3 = 2.5 + c \implies c = 0.5$$

$$v(t) = 2.5 + 0.5e^{-10t}.$$

$$V_R(t) = 5 - v(t) = 2.5 - 0.5e^{-10t}.$$

As $V_R(0) = 2.5 - 0.5 = 2$ and as $t \to \infty$, $V_R(t) \to 2.5$, hence this matches with the plot drawn using phase portrait.

(g) The voltage source is now precisely set to give a sinusoidal voltage $V(t) = \sin 5t$. Given that initially no current is passing through the circuit, solve the differential equation using an integrating factor and find the voltage signal $V_R(t)$, evaluating any constants involved.

$$\frac{dv(t)}{dt} + v(t)\frac{R}{L} = \frac{V(t)}{L}$$

$$\frac{dv(t)}{dt} + 10i(t) = 5\sin 5t$$

$$\mu\frac{dv}{dt} + 10\mu v = 5\mu\sin 5t$$

Integrating factor $\mu = e^{\int 10dt} = e^{10t}$.

$$e^{10t}\frac{dv}{dt} + 10e^{10t}v = 5e^{10t}\sin 5t$$

$$\frac{d}{dt}(e^{10t}v) = 5e^{10t}\sin 5t$$

$$e^{10t}v = \int 5e^{10t}\sin 5t\ dt$$

$$e^{10t}v = \frac{e^{10t}}{5}(2\sin 5t - \cos 5t) + c$$

$$v(t) = \frac{1}{5}(2\sin 5t - \cos 5t) + ce^{-10t}$$

Using the given initial condition $i(0) = 0 \implies v(0) = V(0) - I(0)R = \sin 0 - 0 = 0$,

$$v(0) = \frac{1}{5}(2\sin 0 - \cos 0) + c \implies c = \frac{1}{5}$$

$$v(t) = \frac{1}{5}(2\sin 5t - \cos 5t + e^{-10t})$$

$$V_R(t) = \sin 5t - v(t) = \frac{1}{5}(3\sin 5t + \cos 5t - e^{-10t})$$

(h) Imagine that this circuit is placed somewhere unattended and a toddler grabs this opportunity and starts monkeying with the circuit. She sets the voltage source at an arbitrary value of $V(t) = 9.78 + 6.54\sin(5t)$. Only using your solutions to (d) and (e), find $V_R(t)$ for this case. Be careful with the constants involved because now we do not know the initial current in the circuit.

For input $V(t) = 5$, output $V_R(t) = 2.5 - c_1e^{-10t}$. Because the system is linear, for input $V(t) = 9.78$, output $V_R(t) = 9.78(2.5 - c_1e^{-10t})$.
Similarly, because for input $V(t) = \sin 5t$, output $V_R(t) = \frac{1}{5}(3\sin 5t + \cos 5t - c_2e^{-10t})$, if input is $V(t) = 6.54\sin 5t$, output becomes $V_R(t) = \frac{6.54}{5}(3\sin 5t + \cos 5t - c_2e^{-10t})$. Now for input $V(t) = 9.78 + 6.54\sin 5t$, the output

$$V_R(t) = 9.78(2.5 - c_1e^{-10t}) + \frac{6.54}{5}(3\sin 5t + \cos 5t - c_2e^{-10t})$$

$$V_R(t) = 24.45 + 1.308(3\sin 5t + \cos 5t) + (-1.308c_2 - 9.78c_1)e^{-10t}$$

$$V_R(t) = 24.45 + 1.308(3\sin 5t + \cos 5t) + ce^{-10t}$$

The constant c cannot be evaluated because we do not know the initial state of the circuit.

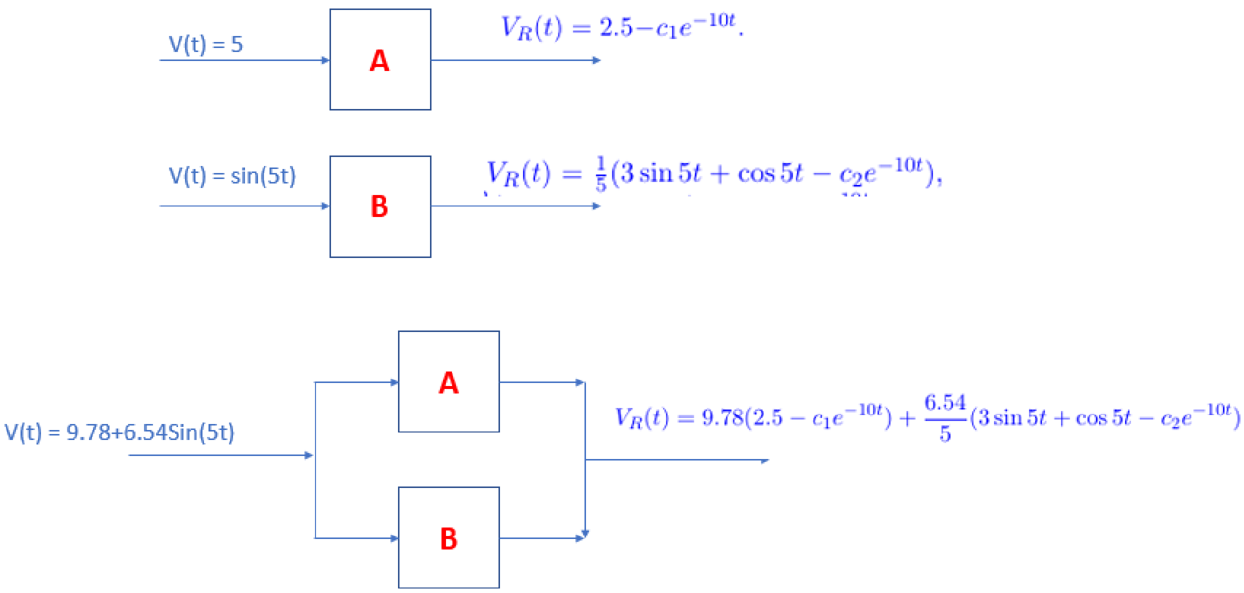


Figure 10: Block Diagram of a LinearSystem