MT232: Differential Equations

Homework 1 Solution

Due Fri Sep 28, 9:00 AM *Fall 2018*

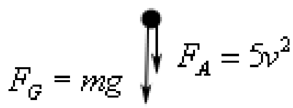
Problem 1

Consider a cannon that is placed on a bridge that has a height of 100 m. A 40 kg cannon-ball is shot vertically upward with a speed of 10 ms^{-1}. The air resistance at any instant is $5v^2$ N opposite to the direction of motion, where v is the velocity of the ball. (Beware that squaring v loses its sign).

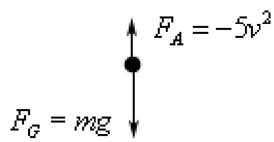
(a) What forces are acting on the ball during the upward and the downward motion? Also draw separate free-body diagrams that identify the forces acting on the object and their directions during the upward and downward motion.

The downward direction is considered positive for this question.

(i) upward motion



(ii) downward motion



(b) We are interested in velocity $v(t)$ of the ball. Write down the initial value problems (IVP) for the

(i) upward motion

We know that

$$ma = F_{net} = F_a + F_d$$
$$mv' = mg + 5v^2$$
$$v' = g + \frac{5}{m}v^2$$
$$v' = g + 0.125v^2,\ v(0) = -10$$

(ii) downward motion

$$ma = F_{net} = F_a + F_d$$
$$mv' = mg - 5v^2$$
$$v' = g - \frac{5}{m}v^2$$
$$v' = g - 0.125v^2,\ v(t_0) = 0$$

t_0 is the time when the canon ball is at the highest point and starts the downward motion.

(c) If the air resistance is $10v$ N instead, write down a single IVP for the trajectory of the ball that is valid until the ball hits the ground on its way back. [Hint: In this case, the sign of v can take care of the reversing direction of the air resistance.]

$$ma = F_{net} = F_a + F_d$$

$F_d = -10v$ in both upward and downward motion

$$mv' = mg - 10v$$

$$v' = g - 0.25v, \ v(0) = -10$$

Problem 2

Consider a tank at a chemical industry that has initially 800 gallons of water with 2 ounce of pollution dissolved in it. Polluted water which has 5 ounce/gallon of pollution flows into the tank at a rate of 3 gallons/hour. We assume the solution is mixed thoroughly as soon as the polluted water comes in and a mixed solution leaves the tank at a rate of 3 gallons/hour. Assume the polluted water disperses as soon as it enters the tank to give a uniform concentration in the tank. [Hint: Mixing tank example, page 24 of the textbook.]

(a) Write down the initial value problem for the amount $Q_1(t)$ of pollution in the tank.

$$Q_1'(t) = \text{input rate of pollution} - \text{output rate of pollution} = R_{in} - R_{out}$$

$$R_{in} = (\text{input rate of water})(\text{concentration of pollution in inflow}) = (3)(5)$$

$$R_{out} = (\text{output rate of water})(\text{concentration of pollution in outflow}) = (3)\left(\frac{Q_1(t)}{800}\right)$$

$$Q_1'(t) = (3)(5) - (3)\left(\frac{Q_1(t)}{800}\right), \ Q_1(0) = 2$$

(b) When the amount of pollution in the tank is 500 ounces, the polluted water does not flows in the tank anymore. At this instant, fresh water starts to enter the tank at a rate of 2 gallons/hour and the out flow increase to 4 gallons/hour. Write down the IVP for the amount of pollution $Q_2(t)$ in the tank in this new scenario.

$$Q_1'(t) = \text{input rate of pollution} - \text{output rate of pollution} = R_{in} - R_{out}$$

$$R_{in} = (\text{input rate of water})(\text{concentration of pollution in inflow}) = (2)(0) = 0$$

$$R_{out} = (\text{output rate of water})(\text{concentration of pollution in outflow}) = (4)\left(\frac{Q_2(t)}{800 - 2(t - tm)}\right)$$

t_m is the time when the quantity of pollution reaches 500 ounces.

$$Q_2'(t) = -4\left(\frac{Q_2(t)}{800 - 2(t - t_m)}\right), \ Q_2(t_m) = 500$$

Problem 3

Recall the population dynamics problem in class in which the change in population was modeled by the following equation

$$\frac{dP}{dt} = kP.$$

(a) Consider the population of a country when people can immigrate into the country at a constant rate r. Derive a differential equation for the population $P(t)$ when $r > 0$.

$$\frac{dP}{dt} = kP + r$$

(b) Determine the differential equation for the population of the country when people are allowed to emigrate from the country at a constant $r > 0$.

$$\frac{dP}{dt} = kP - r$$

(c) The given model does not account for the death rate and birth rate. We can use net rate, the difference between the birth rate and the death rate, to model the changing population of a community. Determine a model for the population $P(t)$ when the birth rate and death rate are proportional to the population present at time t.

$$\frac{dP}{dt} = k_1 P - k_2 P$$

$k_1 > 0$ is the birth rate while $k_2 > 0$ is the death rate

(d) Determine a model for $P(t)$ when the birth rate is proportional to the population at time t but the death rate is proportional to the square of the population at time t. Use the concept of net rate introduced in the last part.

$$\frac{dP}{dt} = k_1 P - k_2 P^2$$

Problem 4

Verify that the given functions are solutions of the corresponding differential equations. Assume an appropriate interval I of definition for each solution. Recall that the solution must be continuous in this interval.

(a) $y'' - 2y' + 5y = 0$; with solution $y(x) = e^x \sin 2x$

$$y' = 2e^x \cos 2x + e^x \sin 2x$$

$$y'' = 4e^x \cos 2x - 3e^x \sin 2x$$

Putting the values in left hand side of equation:

$$(4e^x \cos 2x - 3e^x \sin 2x) - 2(2e^x \cos 2x - e^x \sin 2x) + 5e^x \sin 2x = 0$$

$$4e^x \cos 2x - 4e^x \cos 2x - 5e^x \sin 2x + 5e^x \sin 2x = 0$$

$$0 = 0$$

All the terms cancel out to give 0 which is the right hand side.

(b) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0$; with solution $y(x) = e^{-2x} + e^x$

$$\frac{dy}{dx} = -2e^{-2x} + e^x$$

$$\frac{d^2y}{dx^2} = 4e^{-2x} + e^x$$

Putting the values in left hand side of equation:

$$(4e^{-2x} + e^x) + (-2e^{-2x} + e^x) - 2(e^{-2x} + e^x) = 0$$

$$4e^{-2x} - 4e^{-2x} + 2e^x - 2e^x = 0$$

$$0 = 0$$

(c) A set of two first order differential equations $\left\{\dot{x} = x + 3y,\ \dot{y} = 5x + 3y;\right.$
with solution set $\left\{x(t) = e^{-2t} + 3e^{6t},\ y(t) = -e^{-2t} + 5e^{6t}.\right.$

$$\dot{x} = -2e^{-2t} + 18e^{6t}$$

$$\dot{y} = 2e^{-2t} + 30e^{6t}$$

Putting the values of x and y in right hand side of $\dot{x} = x + 3y$ gives:

$$\dot{x} = e^{-2t} + 3e^{6t} + 3(-e^{-2t} + 5e^{6t})$$

$$\dot{x} = -2e^{-2t} + 18e^{6t}$$

The right hand values comes out to be equal to $\dot{x}$. Putting the values of x and y in right hand side of $\dot{y} = 5x + 3y$ gives:

$$\dot{y} = 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t})$$

$$\dot{y} = 2e^{-2t} + 30e^{6t}$$

The right hand values comes out to be equal to $\dot{y}$.

Problem 5

Some initial value problems are given below. For each of the following,

(a) Identify the dependent and independent variables

(b) Determine whether the differential equation is linear or nonlinear

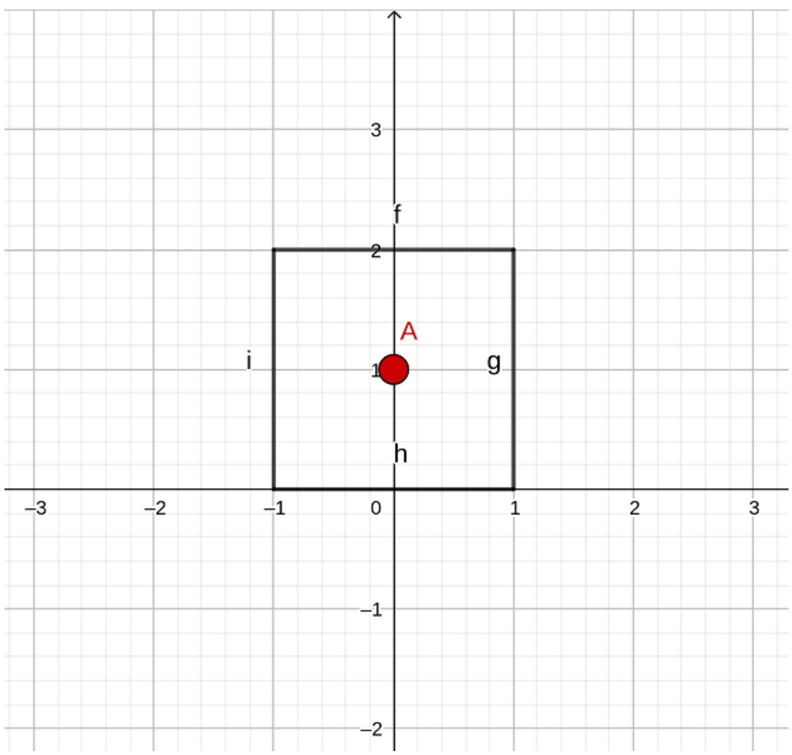
(c) Explain if you can guarantee a unique solution using existence and uniqueness theorem

(i) $y' = 4 + y^3, \;\; y(0) = 1$

(a) Dependent variable: y
Independent Variable: not mentioned in the equation (we assume the independent variable is x for this question)

(b) The differential equation is non-linear due to y^3 term

(c) $f(x, y) = y' = 4 + y^3$ is continuous at $(0, 1)$.
$\dfrac{\partial f}{\partial y} = 3y^2$ is continuous at $(0, 1)$. So we can make a rectangle around this point in which both functions are continuous. A sample rectangle is shown in the figure. Hence, the existence and



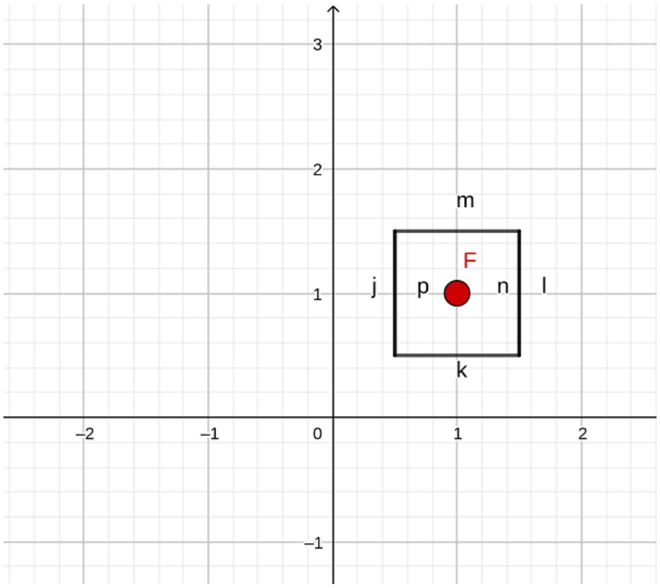
uniqueness theorem guarantees a unique solution somewhere in the interval $x \in [-1, 1]$ for which we have drawn the rectangle.

(ii) $y' = \sqrt{y}, \ y(1) = 1$

(a) Dependent variable: y Independent Variable: not mentioned in the equation (we assume the independent variable is x for this question)

(b) The differential equation is non-linear due to $\sqrt{y}$ term

(c) $f(x, y) = y' = \sqrt{y}$ is continuous at $(1, 1)$.
$\dfrac{\partial f}{\partial y} = \dfrac{1}{2\sqrt{y}}$ is continuous at $(1, 1)$. So we can make a rectangle around this point in which both functions are continuous. A sample rectangle is shown in the figure. Hence, the existence and uniqueness theorem guarantees a unique solution somewhere in the interval $x \in [0.5, 1.5]$ for which we have drawn the rectangle.

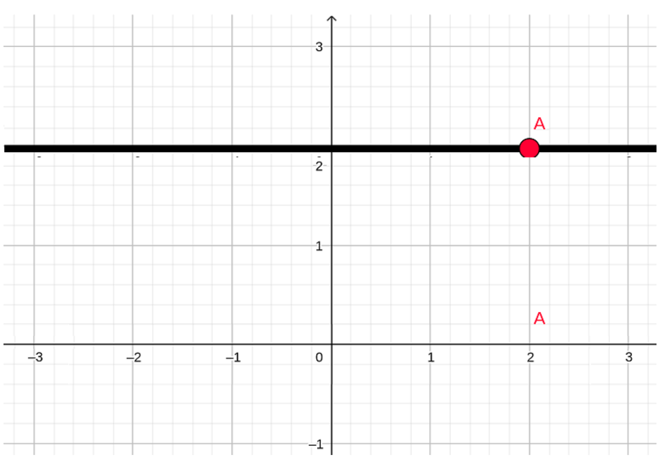


(iii) $\dot{x} = \dfrac{t}{x-2}, \ x(0) = 2$

(a) Dependent variable: x, independent variable: t

(b) The differential equation is non-linear due to $(x - 2)^{-1}$ term

(c) $f(t, x) = \dot{x} = \dfrac{t}{x-2}$ is non-continuous at $(0, 2)$. In fact it's discontinuous everywhere at $x = 2$. So we cannot make any rectangle around this point in which both functions are continuous. Hence, the existence and uniqueness theorem does not tell whether a unique solution exists or not.



(iv) $y' = x \tan y, \quad y(0) = 0$

(a) Dependent variable: y

Independent variable: x

(b) The differential equation is non-linear due to $\tan y$ term

(c) $f(x, y) = y' = x \tan y$ is continuous at $(0, 0)$.

$\dfrac{\partial f}{\partial y} = x \sec^2 y$ is continuous at $(0, 0)$.

So we can make a rectangle around this point in which both functions are continuous. A sample rectangle is shown in the figure. Hence, the existence and uniqueness theorem guarantees a unique solution somewhere in the interval $x \in [-0.5, 0.5]$ for which we have drawn the rectangle.