MT232: Differential Equations

Homework 6

Due Fri Dec 21, 9:00 AM

Fall 2018

Tips to avoid plagiarism

- Do not copy the solutions of your classmates.
- Your are encouraged to discuss the problems with your classmates in whatever way you like but make sure to REPRODUCE YOUR OWN SOLUTIONS in what you submit for grading.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1

(a) Determine whether Laplace transform of given function exist? If yes then, find the range of values of s for which the Laplace transform $\mathcal{L}\{f(t)\}$ exists.

$$f(t) = t \sin t$$

(b) Consider the differential equation

$$y''(t) + 4y'(t) + 8y(t) = f(t)$$

where $f(t)$ is the input or forcing/driving function. Evaluate the following using Laplace Transform method and sketch their graphs.

(i) Impulse response (i.e. $f(t)$ is unit impulse, assuming zero initial conditions)

(ii) Step response (i.e. $f(t)$ is unit step, assuming zero initial conditions)

(iii) when $f(t) = e^{-2t}$, given $y(0) = 2$ and $y'(0) = 1$.

Problem 2

Figure 1 shows a cart of mass m attached to a fixed spring with stiffness k and a dashpot with damping coefficient b. The cart is shown in its equilibrium position but it can have a displacement x towards right or left causing an extension or compression in the spring. Given that $m = 1$ kg and $k = 4$ Nm^{-1}.

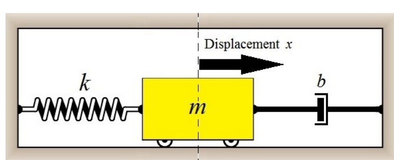

Figure 1: Spring mass system connected to a dashpot.

The cart is pulled 1 m to the right and released from rest.

(a) Write down a linear second order differential equation for this system, in terms of m, b and k.

(b) Write down the differential equation in terms of ζ and ω_o.

(c) Without solving the differential equation, find the range of values of the damping coefficient b for which the cart oscillates but the amplitude of its oscillations decays to 13.5% within the first 2 seconds of release. What range of values of quality factor Q does this correspond to?

Consider $b = 2$ kgs^{-1} for rest of this problem. Now consider a new initial condition in which at $t = 0$, the cart passes its equilibrium position at a speed of 1 ms^{-1} towards right. At $t = 0$, a constant force of 4 N towards right is also applied to the cart and is maintained for all $t > 0$.

(d) Write down the differential equation of the system with an appropriate forcing function.

(e) With these values of b and the force applied, the cart is expected to oscillate before approaching a position x_∞ as $t \to \infty$. Using a method of your choice, solve the differential equation and find the value of x_∞.

(f) Sketch the graph of $x(t)$ for $t \geq 0$.

(g) Evaluate how far does the cart go past the point x_∞ for the first time during oscillation. (Look at your graph if you are having trouble understanding this part.)

Problem 3

Suppose compartments A and B shown in Fig.3 are filled with fluids and are separated by a permeable membrane. The figure is a compartmental representation of the exterior and interior of a cell. Suppose, too, that a nutrient necessary for cell growth passes through the membrane.

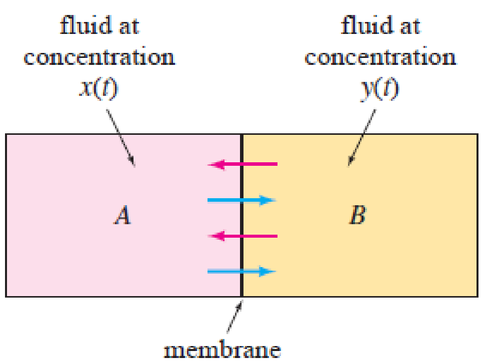

Figure 2: Nutrient flow through a membrane

A model for the concentrations $x(t)$ and $y(t)$ of the nutrient in compartments A and B, respectively, at time t is given by the linear system of differential equations

$$x'(t) = \frac{k}{V_A}(y - x)$$

$$y'(t) = \frac{k}{V_B}(x - y)$$

where V_A and V_B are the volumes of the compartments, and $k > 0$ is a permeability factor. Let $x(0) = x_0$ and $y(0) = y_0$ denote the initial concentrations of the nutrient. Solely on the basis of the equations in the system and the assumption $x_0 = 1$, $y_0 = 2$.

(a) Find its general solution for $V_A = V_B = 2$ m^3 and $k = 1$

(b) Sketch its general phase-portrait of the system, and show at least 8 trajectories in the phase-plane for $t \in (-\infty, \infty)$ starting from different initial conditions of your choice all over the plane (this is how we made those exotic sketches in class).

(c) Classify its equilibrium point as node, saddle, ellipse or spiral, and also comment on its stability.

(d) Now for $t \in (0, \infty)$, sketch the trajectory of its solution in the phase-plane that starts from the given initial condition at $t = 0$.

(e) Now use the given initial conditions to find its solution (i.e. evaluate c_1 and c_2).

(f) Initially which of the concentrations $x(t)$ or $y(t)$ of the nutrient changes fast.

Problem 4

(a) A patient died during his treatment in a hospital. His relatives are putting the blame on a nurse who gave the patient an overdose of a drug Mathemine. But the nurse claims that although she did give a slight overdose but the amount she injected was still safe.

And this is nurse Jenkins...who we'll be blaming if the hospital kills your husband...

When Mathemine is injected into the muscle, it gradually flows into the bloodstream and mixes with the blood. Then over time, the body slowly gets rid of the drug through excretion. Let $x(t)$ be the amount of Mathemine in the blood at time t, and $y(t)$ be its amount in the muscle. The relationship between these quantities is given by the differential equations

$$x'(t) = -3x(t) + 2y(t)$$

$$y'(t) = x(t) - 2y(t)$$

Initially there was zero Mathemine in the patient's blood when the nurse injected 12 mg of Mathemine into his muscle. If the amount of Mathemine in the blood at a certain instance rises above 6 mg, it is lethal. Find out whether or not the patient died due to drug overdose and if the nurse is responsible for the death. (Hint: Find $x(t)$ by solving the system of differential equations and find out whether its value rises above 6 or not)

(b) In an $n \times n$ linear system of differential equations with constant coefficients

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),$$

prove that if matrix $\mathbf{A}$ is non-singular (invertible), any value of velocity vector is achievable at some point in the phase plane.

Problems 5

Tank T_1 and T_2 in Fig. 3 contain initially 100 gallons of water each. In T_1 the water is pure, whereas 180 kg of fertilizer are dissolved in T_2. By circulating liquid at a rate of 2 gal/min and stirring (to keep the mixture uniform) the amounts of fertilizer $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t.

(a) Write the differential equations that describes the above mathematical model.

(b) Find the general solution of the above mathematical model.

(c) Sketch the general phase portrait of the system.

(d) How long should we let the liquid circulate so that T_1 will contain at least half as much fertilizer as there will be left in T_2?

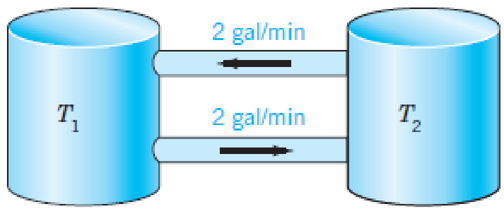

Figure 3: System of tanks

Problem 6

Consider the differential equation

$$y''(t) + 7y'(t) + 12y(t) = te^{-3t},$$

with initial conditions $y(0) = 0$ and $y'(0) = 1$.

(a) Convert the differential equation into a linear system of first order differential equations.

(b) Find the complementary solution of the system.

(c) Find the particular solution of the system using variation of parameters.

(d) Determine the solution $y(t)$ of the differential equation.

THE END
