MT232: **Differential Equations**

Homework 4

Due Fri Nov 23, 9:00 AM

Fall 2018

Tips to avoid plagiarism

- Do not copy the solutions of your classmates.
- Your are encouraged to discuss the problems with your classmates in whatever way you like but make sure to REPRODUCE YOUR OWN SOLUTIONS in what you submit for grading.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1

For each of the following initial and boundary value problems,

(a) $y'' + 4y' + 5y = 0, \quad y(1) = 3$ and $y'(1) = 9$

(b) $y'' - 4y' + 4y = 0, \quad y(1) = 1$ and $y'(1) = 1$

(c) $y'' + 5y' + 6y = 0, \quad y(0) = 1$ and $y'(0) = 0$

(d) $4y'' + y' - 4y = 0, \quad y(0) = 0$ and $y'(0) = 1$

(e) $y'' + y' - 6y = 0, \quad y(0) = 10$ and $y'(0) = 0$

(i) Determine whether system is stable, marginally stable or unstable.

(ii) For stable and marginally stable systems, find value of natural frequency (ω_o), damping ratio (ζ) and quality factor (Q) .

(iii) Is the system undamped, underdamped, overdamped or critically damped?

(iv) In case of underdamped system, find the pesudo-natural frequency of the system.

(v) Solve the differential equation for the given initial or boundary conditions. In case of sinusoidal solutions, write the solution in $A\cos(wt - \phi)$ form.

(vi) Sketch the graph of the solution.

Problem 2

Figure 1 shows a cart of mass m attached to a fixed spring with stiffness k. The cart is shown in its equilibrium position but it can have a displacement x towards right or left causing a resulting extension or compression in the spring. Given that $m = 3$ kg and $k = 12$ Nm^{-1}, use these values for all parts of this problem unless stated otherwise.

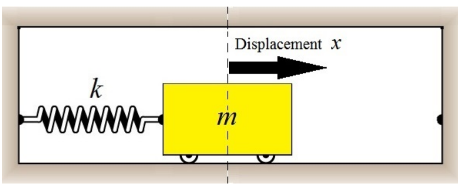


Figure 1: Ideal spring mass system with no friction or damping

(a) Write down a linear second order differential equation for the system, in terms of m, k, and x.

(b) The cart is pulled 0.5 m to the right and released at $t = 0$. It is expected to oscillate about its equilibrium position with a natural frequency ω_0.

  (i) Find $x(t)$ for $t \geq 0$ by solving the differential equation. What is the natural frequency ω_0?

  (ii) Determine damping and stability condition of system.

  (iii) Sketch the graph of $x(t)$, clearly showing the amplitude and time period of the oscillation.

Suppose that now a dashpot with damping coefficient b is attached to the cart as shown in Figure 2. We have three dashpots of three different damping coefficients available. We attach the dashpots one by one and examine the effect of each dashpot on the motion of the cart.

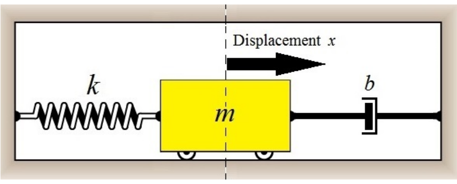


Figure 2: Spring mass system connected to a dashpot attached for damping

(a) Write down a linear second order differential equation for this system, in terms of m, b, k and x.

(b) First, the dashpot with $b = 6$ kgs^{-1} is attached to the cart.

  (a) Determine whether system is stable, marginally stable or unstable.

  (b) For stable and marginally stable system, find value of (ω_o) and damping ratio (ζ).

  (c) What is the damping condition of system (undamped, underdamped, overdamped or critically damped).

  (d) The cart is once again pulled 0.5 m to the right and released at $t = 0$. Find $x(t)$ for $t \geq 0$ by solving the differential equation and sketch the graph of $x(t)$.

(c) Repeat part (b) for the second dashpot with $b = 12$ kgs^{-1}.

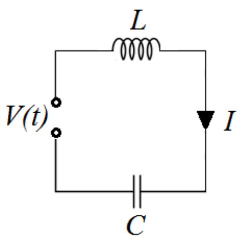
(d) Repeat part (b) for the third dashpot with $b = 24$ kgs^{-1}.

(e) Comment on the response of $x(t)$ find in part (b,c and d). Explain your answer.

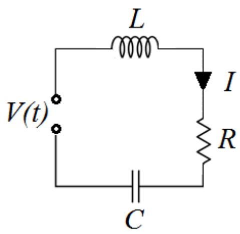
(f) Which $x(t)$ will settle in minimum time?

Problem 3

In this problem you will understand the concept of tuning a circuit to a frequency of the received signal, which can be the frequency of a radio station or the frequency at which your mobile service provider transmits.



(a) LC circuit



(b) RLC circuit

Figure 3

(a) Figure 3(a) shows an LC circuit driven by a variable voltage source $V(t)$, which causes a current $I(t)$ to flow through the circuit.

(i) From your notes, write down the differential equation for current I in the circuit, in terms of L, C and V. Convert the equation into its standard form.

(ii) Write down the characteristic equation $S(r)$ for the differential equation and find its roots in terms of L, C and i.

(iii) Using your answer to (ii), find the natural frequency ω_0 of the circuit?

(iv) Rewrite the differential equation in terms of ω_0, instead of L and C.

(v) Let $V(t) = \cos \omega t$, where ω is the driving frequency. At what frequency ω does the resonance happen, i.e. the amplitude of current $I(t)$ grows to a maximum value?

(vi) Given $L = 1$ H and $C = 0.125$ F, calculate the value of the resonant frequency.

(vii) Find the expression for particular solution of current $I(t)$ in the circuit at resonant frequency, and sketch its graph.

(b) Figure 3(b) shows an RLC circuit driven by a variable voltage source $V(t)$, which causes a current $I(t)$ to flow through the circuit. By varying the values of R, L or C, this circuit can be tuned to the frequency of a signal received as voltage $V(t)$.

(i) From your notes, write down the differential equation for current I in the circuit, in terms of R, L, C and V. Convert the equation into its standard form.

(ii) Using your notes, rewrite this equation in terms of quality factor Q and natural frequency ω_0.

(iii) Write down the characteristic equation $S(r)$ for the differential equation and find its roots in terms of Q, ω_0 and i.

(iv) What is the damped natural frequency ω_d of the circuit, in terms of Q and ω_0?

(v) For $V(t) = \cos \omega t$, $R = 2\ \Omega$, $L = 1$ H and $C = 0.125$ F, evaluate ω_d. Use these values of R, L and C for the remaining parts of this problem.

(vi) Find the steady-state current $I(t)$ in the circuit. (Recall that the steady-state solution is the solution as $t \to \infty$. For stable systems, steady-state solution is equal to the particular solution because the complementary solution approaches 0 as $t \to \infty$.)

(vii) For this circuit, its resonant frequency ω_r is not exactly equal to the damped natural frequency ω_d. Resonance happens at a slightly different frequency. In this part, you will calculate ω_r. Convert $I(t)$ into the form $I(t) = A(\omega) \cos(\omega t - \phi(\omega))$, where $A(\omega)$ and $\phi(\omega)$ are functions of ω. Resonance will happen when the value of $A(\omega)$ is maximized. Find the value of ω for which A is maximized. This is the resonant frequency ω_r for the circuit. (You do not need to evaluate ϕ)

(viii) Suppose the radio signal received as $V(t)$ is at a frequency $f = 2$ Hz (recall that $\omega = 2\pi f$). The RLC circuit is said to be tuned to the received radio signal when the amplitude of current $I(t)$ through the circuit is maximum. This happens when the resonant frequency f_r of the circuit equals 2 Hz, the frequency of the radio signal. The resonant frequency of the circuit can be changed by varying the values of either R, L or C. Suppose that we can only control the value of C. Keeping $R = 2\ \Omega$, $L = 1$ H, calculate the value of C for which the resonant frequency of the circuit $f_r = 2$ Hz and the circuit will be tuned to the received radio signal.
