MT232: Differential Equations

Homework 2

Due Fri Oct 12, 9:00 AM *Fall 2018*

Tips to avoid any cases of plagiarism

- Do not look at the solutions of your classmates.
- Your are encouraged to discuss the homework with your classmates, but restrict yourself to oral discussions only.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1

When certain kinds of chemicals are combined, the reactions are autonomous and the rate at which the new compound is formed is modeled by the differential equation

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X)$$

where $k > 0$ is a constant of proportionality and $\beta > \alpha > 0$. Here $X(t)$ denotes the number of grams of the new compound formed in time t

(a) Use a phase portrait of the differential equation to predict the behavior of $X(t)$ as $t \to \infty$

(b) Consider the case when $\alpha = \beta$. Use a phase portrait of the differential equation to predict the behavior of X(t) as $t \to \infty$ when $X(0) = \alpha > 0$.

(c) Find the explicit solution of the DE in the case when $k = 1$ and $\alpha = \beta = 5$ for

  (i) $X(0) = \frac{\alpha}{2}$

  (ii) $X(0) = 2\alpha$.

  Graph these two solutions.

(d) Does the behavior of the solutions as $t \to \infty$ agree with your answers to part (b)?

Problem 2

Consider a body is falling from the top of the tower of Pisa under the influence of gravity and air resistance. The mass of the body is m, v is its speed, g is the acceleration due to gravity and b is the air drag coefficient.

(a) Derive the DE of the above model. (Note: Take downward direction positive)

(b) Show that the differential equation represents an autonomous system.

(c) Find and classify the equilibrium point of the system and draw its phase portrait.

(d) Now consider that from the top of the tower of Pisa, a ping-pong ball of mass 5 g is being dropped from rest. What happens to the speed of the ball during the fall, considering the coefficient of air drag $b = 0.47$?

(e) In another trial, the ball is shot down at an initial speed of 5 ms^{-1}. Now during the fall, what happens to the speed of the ball?

(f) In yet another trial, a bouncy ball is dropped from rest which is of the same shape and size but 10 times heavier than the ping-pong ball. Does this ball reach the ground faster than the ping-pong ball? Explain.

(g) But if we drop two balls of masses 20 kg and 200 kg from the tower at once, why do they reach the ground almost at the same time, although one is 10 times heavier than the other?

Problem 3

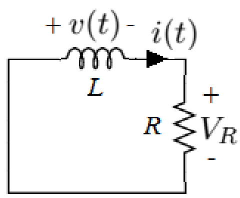
Consider the following differential equation with initial condition $(1, 2)$.

$$x\frac{dy}{dx} + y = 2y^2$$

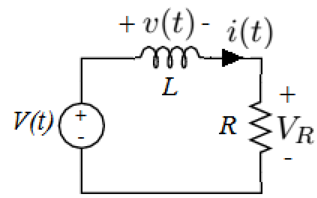
Solve the differential equation using an appropriate method you have studied so far until Oct 5.

Problem 4

Figure 1(a) shows a circuit in which an inductor of inductance L is connected in series with a resistor R. Let's assume that the voltage across inductor and the current flowing through the inductor at time t are $v(t)$ and $i(t)$ respectively. Given that $R = 2\Omega$ and $L = 0.2$H.



(a)



(b)

Figure 1: RL circuits for Problem 4

(a) Using Kirchhoff's Voltage Law, write down the differential equation of voltage $v(t)$ in the circuit given in Figure 1(a).

(b) At $t = 0$, a current of 5 A was passing through the inductor. Without solving the differential equation, sketch V_R for $t = 0$ to $t \to \infty$. Mark clearly on your graph, the values of V_R at $t = 0$ and $t \to \infty$.

(c) For the conditions in (a), guess the exact solution of the differential equation, evaluating any constants involved.

Now a voltage source $V(t)$ is added in series to the circuit as shown in Figure 1(b).

(d) Write down the differential equation of voltage $v(t)$ in the circuit given in Figure 1(b).

(e) The voltage source is set to give a constant DC voltage of $V(t) = 5$ V. Given that initially a current of 1 A is passing through the circuit. Without solving the differential equation, sketch V_R for $t = 0$ to $t \to \infty$. Mark clearly on your graph, the values of V_R at $t = 0$ and $t \to \infty$.

(f) Express this linear differential equation in the standard form discussed in the lecture. For $V(t) = 5$ V, solve the differential equation using an integrating factor and find the voltage signal $V_R(t)$, evaluating any constants involved. Verify that your solution agrees with your plot in (c).

(g) The voltage source is now precisely set to give a sinusoidal voltage $V(t) = \sin 5t$. Given that initially no current is passing through the circuit, solve the differential equation using an integrating factor and find the voltage signal $V_R(t)$, evaluating any constants involved.

(h) Imagine that this circuit is placed somewhere unattended and a toddler grabs this opportunity and starts monkeying with the circuit. She sets the voltage source at an arbitrary value of $V(t) = 9.78 + 6.54 \sin(5t)$. Only using your solutions to (d) and (e), find $V_R(t)$ for this case. Be careful with the constants involved because now we do not know the initial current in the circuit.
