MT232: Differential Equations

Homework 1

Due Fri Sep 28, 9:00 AM

Fall 2018

Tips to avoid any cases of plagiarism

- You must NOT look at the solutions of your classmates.
- Your are encouraged to discuss the homework with your classmates, but restrict yourself to oral discussions only.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1

Consider a cannon that is placed on a bridge that has a height of 100 m. A 40 kg cannon-ball is shot vertically upward with a speed of 10 ms^{-1}. The air resistance at any instant is $5v^2$ N opposite to the direction of motion, where v is the velocity of the ball. (Beware that squaring v loses its sign).

(a) What forces are acting on the ball during the upward and the downward motion? Also draw separate free-body diagrams that identify the forces acting on the object and their directions during the upward and downward motion.

(b) We are interested in velocity $v(t)$ of the ball. Write down the initial value problems (IVP) for the

  (i) upward motion

  (ii) downward motion

(c) If the air resistance is $10v$ N instead, write down a single IVP for the trajectory of the ball that is valid until the ball hits the ground on its way back. [Hint: In this case, the sign of v can take care of the reversing direction of the air resistance.]

Problem 2

Consider a tank at a chemical industry that has initially 800 gallons of water with 2 ounce of pollution dissolved in it. Polluted water which has 5 ounce/gallon of pollution flows into the tank at a rate of 3 gallons/hour. We assume the solution is mixed thoroughly as soon as the polluted water comes in and a mixed solution leaves the tank at a rate of 3 gallons/hour. Assume the polluted water disperses as soon as it enters the tank to give a uniform concentration in the tank. [Hint: Mixing tank example, page 24 of the textbook.]

(a) Write down the initial value problem for the amount $Q_1(t)$ of pollution in the tank.

(b) When the amount of pollution in the tank is 500 ounces, the polluted water does not flows in the tank anymore. At this instant, fresh water starts to enter the tank at a rate of 2 gallons/hour and the out flow increase to 4 gallons/hour. Write down the IVP for the amount of pollution $Q_2(t)$ in the tank in this new scenario.

Problem 3

Recall the population dynamics problem in class in which the change in population was modeled by the following equation

$$\frac{dP}{dt} = kP.$$

(a) Consider the population of a country when people can immigrate into the country at a constant rate r. Derive a differential equation for the population $P(t)$ when $r > 0$.

(b) Determine the differential equation for the population of the country when people are allowed to emigrate from the country at a constant $r > 0$.

(c) The given model does not account for the death rate and birth rate. We can use net rate, the difference between the birth rate and the death rate, to model the changing population of a community. Determine a model for the population $P(t)$ when the birth rate and death rate are proportional to the population present at time t.

(d) Determine a model for $P(t)$ when the birth rate is proportional to the population at time t but the death rate is proportional to the square of the population at time t. Use the concept of net rate introduced in the last part.

Problem 4

Verify that the given functions are solutions of the corresponding differential equations. Assume an appropriate interval I of definition for each solution. Recall that the solution must be continuous in this interval.

(a) $y'' - 2y' + 5y = 0$; with solution $y(x) = e^x \sin 2x$

(b) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0$; with solution $y(x) = e^{-2x} + e^x$

(c) A set of two first order differential equations $\left\{\dot{x} = x + 3y,\ \dot{y} = 5x + 3y;\right.$
with solution set $\left\{x(t) = e^{-2t} + 3e^{6t},\ y(t) = -e^{-2t} + 5e^{6t}.\right.$

Problem 5

Some initial value problems are given below. For each of the following,

(a) Identify the dependent and independent variables

(b) Determine whether the differential equation is linear or nonlinear

(c) Explain if you can guarantee a unique solution using existence and uniqueness theorem

(i) $y' = 4 + y^3,\ \ y(0) = 1$

(ii) $y' = \sqrt{y},\ \ y(1) = 1$

(iii) $\dot{x} = \dfrac{t}{x-2},\ \ x(0) = 2$

(iv) $y' = x \tan y,\ \ y(0) = 0$