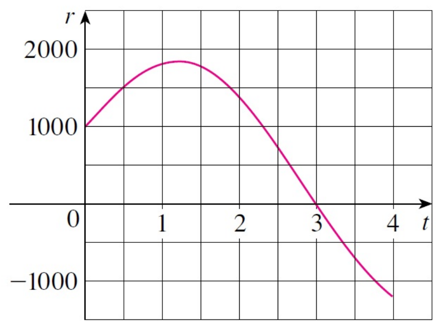
MT110: Calculus and Analytic Geometry

Homework 6 Solution

Due: Fri, Dec 21, 2:00 pm *Fall 2018*

Problem 1

Water flows into and out of a storage tank. A graph of the rate of change $r(t)$ of the volume of water in the tank, in liters per day, is shown in the following. If the amount of water in the tank at time $t = 0$ is 25, 000 liters.



(a) Using the trend in the graph of $r(t)$, describe how the amount of water in the tank is changing from: $t = 0$ to 1.25, $t = 1.25$ to 3, and $t = 3$ to 4. Be careful with the fact that the graph is not of the amount of water. It is for the rate of change of the amount of water.

Solution:

(a) $t = 0$ to $t = 1.25$

Since net rate is positive and increasing, the inflow of water is increasing over time.

(b) $t = 1.25$ to $t = 3$

Since net rate is positive but decreasing, the inflow of water is decreasing over time.

(c) $t = 3$ to $t = 4$

Since net rate is negative and decreasing, the outflow of water is decreasing over time.

(b) Estimate the amount of water in the tank four days later.

Solution:

By the Net Change Theorem, the amount of water after four days is

$$\text{Amount of water} = 25,000 + \int_0^4 r(t)dt$$

Using midpoint rule with $n = 4$ i.e. $\Delta t = \dfrac{4-0}{4} = 4$

$$\int_0^4 r(t) \approx 4[r(0.5) + r(1.5) + r(2.5) + r(3.5)]$$

$$\int_0^4 r(t) \approx 4[1500 + 1770 + 740 + (-690)] = 3320$$

$$\text{Amount of water} = 25,000 + 3320 = 28,320 \text{ liters}$$

(c) Find an upper bound of the error in your estimate.

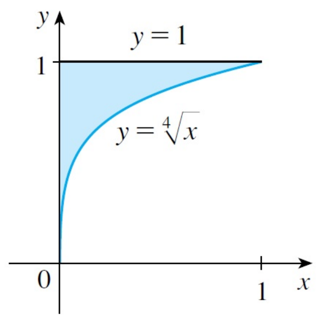
A rough upper bound of error = number of squares the curve passes through $\times$ area of one square $= 12 \times 250 = 3000$

So, with error information, the estimate of the amount of water can be written as

$$28320 \pm 1500 \text{ liters}$$

Problem 2

Find the exact value of the shaded area shown in the following figure.



Solution:

Method I

The shaded area can be written as

$$\text{Shaded area} = \underbrace{\int_0^1 1dt}_{\text{Area of square}} - \underbrace{\int_0^1 \sqrt[4]{x}dt}_{\text{Area under the curve}}$$

$$\text{Shaded area} = x - \left.\frac{4x^{\frac{5}{4}}}{5}\right|_0^1 = 1 - \frac{4}{5} = \frac{1}{5}$$

Method II

$$y = \sqrt[4]{x} \implies x = y^4$$

integrating over y-axis as

$$\text{Shaded Area} = \int_0^1 y^4 dy = \frac{1}{5}$$

Problem 3

If $f(x)$ is continuous on $[a, b]$,

(a) Show that

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

Hint: $-|f(x)| \leq f(x) \leq |f(x)|$

Solution:

Since $-|f(x)| \leq f(x) \leq |f(x)|$ then using property 7 (page 381), it follows that

$$\int_a^b -|f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$$

$$\implies \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

(b) Use the part (a) to show that

$$\left|\int_0^{2\pi} f(x)\sin(2x)dx\right| \leq \int_0^{2\pi} |f(x)|dx$$

Solution:

$$\begin{aligned} \left|\int_0^{2\pi} f(x)\sin(2x)dx\right| &\leq \int_0^{2\pi} |f(x)\sin(2x)|\,dx && \because \text{using part (a)} \\ \left|\int_0^{2\pi} f(x)\sin(2x)dx\right| &\leq \int_0^{2\pi} |f(x)||\sin(2x)|\,dx \\ \implies \left|\int_0^{2\pi} f(x)\sin(2x)dx\right| &\leq \int_0^{2\pi} |f(x)|dx && \because |\sin(2x)| \leq 1 \end{aligned}$$

Problem 4

Use the Fundamental Theorem of Calculus part 1 to find the following derivatives.

(a)

$$\frac{d}{dx}\int_0^{\tan x} \sqrt{t + \sqrt{t}}dt$$

Solution:

Since $f(t) = \sqrt{t + \sqrt{t}}$ is continuous and using the chain rule along with FOTC part 1. Also Let $u = \tan(x)$

$$\begin{aligned} \frac{d}{dx} &= \frac{d}{du}\frac{du}{dx} \\ &= \sqrt{u + \sqrt{u}}\frac{du}{dx} \\ \frac{d}{dx}\int_0^{\tan x} \sqrt{t + \sqrt{t}} &= \sqrt{\tan(x) + \sqrt{\tan(x)}}\frac{d}{dx}[\tan(x)] \\ &= \sqrt{\tan(x) + \sqrt{\tan(x)}}\left(\sec^2(x)\right) \end{aligned}$$

(b)

$$\frac{d}{dx}\int_{1-3x}^{x^2}\frac{u^3}{1+u^2}du$$

Solution:

$$\frac{d}{dx}\int_{1-3x}^{x^2}\frac{u^3}{1+u^2}du = \frac{d}{dx}\left(\int_{1-3x}^{a}\frac{u^3}{1+u^2}du + \int_{a}^{x^2}\frac{u^3}{1+u^2}du\right)$$
$$\frac{d}{dx}\int_{1-3x}^{x^2}\frac{u^3}{1+u^2}du = \frac{d}{dx}\left(-\int_{a}^{1-3x}\frac{u^3}{1+u^2}du + \int_{a}^{x^2}\frac{u^3}{1+u^2}du\right)$$
$$\frac{d}{dx}\int_{1-3x}^{x^2}\frac{u^3}{1+u^2}du = \frac{d}{dx}\left(-\int_{a}^{1-3x}\frac{u^3}{1+u^2}du\right) + \frac{d}{dx}\left(\int_{a}^{x^2}\frac{u^3}{1+u^2}du\right) \quad \because a \in \mathbb{R}$$

Using the FOTC part 1 as function $f(u) = \dfrac{u^3}{1+u^2}$ is continuous and using the chain rule for $t = 1-3x$ and $s = x^2$

$$\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx}$$
$$\frac{d}{dx}\left(-\int_{a}^{1-3x}\frac{u^3}{1+u^2}du\right) = \frac{t^3}{1+t^2}\frac{dt}{dx} = \left(-\frac{(1-3x)^3}{1+(1-3x)^2}\right)(-3)$$
$$\frac{d}{dx} = \frac{d}{ds}\frac{ds}{dx}$$
$$\frac{d}{dx}\left(\int_{a}^{x^2}\frac{u^3}{1+u^2}du\right) = \frac{s^3}{1+s^2}\frac{dt}{dx} = \left(\frac{(x^2)^3}{1+(x^2)^2}\right)(2x)$$

Combining both

$$\frac{d}{dx}\int_{1-3x}^{x^2}\frac{u^3}{1+u^2}du = \left(-\frac{(1-3x)^3}{1+(1-3x)^2}\right)(-3) + \left(\frac{(x^2)^3}{1+(x^2)^2}\right)(2x)$$
$$\frac{d}{dx}\int_{1-3x}^{x^2}\frac{u^3}{1+u^2}du = \frac{3(1-3x)^3}{1+(1-3x)^2} + \frac{2x^7}{1+x^4}$$

Problem 5

For each of the following definite integrals,

(a) $I = \displaystyle\int_0^{\pi} \sin x \sec^2(\cos x)dx$

(i) Find the bounds of the integral using an appropriate property.
Solution:In order to use the property 8 (page 381), we need to find the global extremum of function

$$f(x) = \sin x \sec^2(\cos x), \quad 0 \le x \le \pi$$

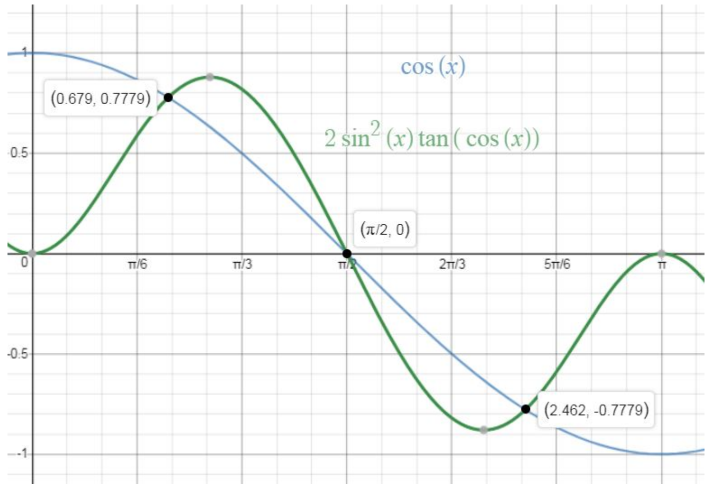
$$f'(x) = \cos(x)\left(\sec^2(\cos(x))\right) - 2\left(\sin^2(x)\right)\left(\tan(\cos(x))\right)\left(\sec^2(\cos(x))\right) = 0$$

$$\implies \sec^2(\cos(x))(\cos(x) - 2\sin^2(x)\tan(\cos(x))) = 0$$

Since $\sec^2(\cos(x)) \neq 0$

$$(\cos(x) - 2\sin^2(x)\tan(\cos(x))) = 0$$

Critical points are $x = 0.679$, $x = \pi/2$, $x = 2.462$



x	0	0.679	$\pi/2$	2.462	π
$f(x)$	0	1.238	1	1.238	0
	m=Abs Min	M=Abs Max			

$$0(\pi - 0) \leq \int_0^\pi \sin x \sec^2(\cos x)dx \leq 1.238(\pi - 0)$$
$$0 \leq \int_0^\pi \sin x \sec^2(\cos x)dx \leq 1.238\pi$$

(ii) First evaluate the indefinite form of the integral using an appropriate substitution.
Solution:Let $u = \cos(x) \implies du = -\sin(x)$

$$I = \int -sec^2(u)du = -\tan(u) + C \implies I = -\tan(\cos(x)) + C$$

(iii) Now evaluate the definite integral.
Solution:

$$I(\pi) - I(0) = -\tan(\cos(\pi)) + \tan(\cos(0)) = 2\tan(1) = 3.114$$

(iv) Using your answer to (iii), evaluate the integral from $-\pi$ to π.
Solution:

$$f(-x) = \sin(-x)\sec^2(\cos(-x) = -\sin x \sec^2(\cos x)) = f(x)$$

since $f(x)$ is odd function then $\int_{-\pi}^{\pi} f(x)dx = 0$

(v) Express the integral as an integral function $\int_a^x f(t)dt$, where a and $f(t)$ are to be specified.

$$I(x) = \int_0^x \sin t \sec^2(\cos t)dt$$

(b) $J = \int_0^{\pi} e^{\cos x} \sin 2x \; dx$

(i) Find the bounds of the integral using an appropriate property.

Solution:In order to use the property 8 (page 381), we need to find the global extremum of function

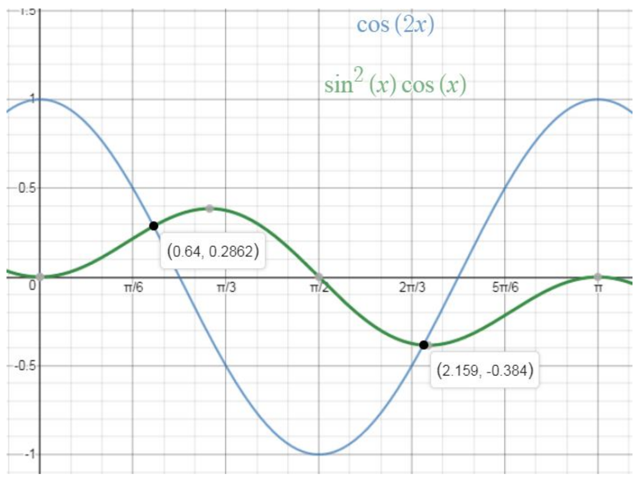
$$f(x) = e^{\cos x} \sin 2x, \quad 0 \le x \le \pi$$

$$f'(x) = 2e^{\cos(x)} \left(\cos(2x) - \sin^2(x)\cos(x)\right) = 0$$

Since $e^{\cos(x)} \neq 0$

$$\left(\cos(2x) - \sin^2(x)\cos(x)\right) = 0$$

Critical points are $x = 0.640$, $x = 2.159$,



x	0	0.640	2.159	π
$f(x)$	0	2.13658	-0.530072	0
		M=Abs Max	m=Abs Min	

$$(-0.53)(\pi - 0) \le \int_0^{\pi} e^{\cos x} \sin 2x dx \le 2.13(\pi - 0)$$

$$-0.53\pi \le \int_0^{\pi} e^{\cos x} \sin 2x dx \le 2.13\pi$$

(ii) First evaluate the indefinite form of the integral using an appropriate substitution.

Solution:

$$\int e^{\cos(x)} \sin(2x)dx = 2\int e^{cos(x)} \sin(x)\cos(x)dx$$

Let $u = \cos(x) \implies du = -\sin(x)$

$$2\int e^{cos(x)} \sin(x)\cos(x)dx = -2\int ue^u du$$

using integration by parts

$$J = -2\int ue^u du = 2e^u(1-u) + C = 2e^{\cos(x)}(1 - \cos(x)) + C$$

(iii) Now evaluate the definite integral.

Solution:

$$J(\pi) - J(0) = 2e^{\cos(\pi)}(1 - \cos(\pi)) - 2e^{\cos(0)}(1 - \cos(0)) = \frac{4}{e}$$

(iv) Using your answer to (iii), evaluate the integral from $-\pi$ to π.

Solution:

$$f(-x) = e^{\cos(-x)}\sin(-2x) = -e^{\cos(x)}\sin(2x) = f(x)$$

since $f(x)$ is odd function then $\int_{-\pi}^{\pi} f(x)dx = 0$

(v) Express the integral as an integral function $\int_a^x f(t)dt$, where a and $f(t)$ are to be specified.

$$J(x) = \int_0^x e^{\cos t}\sin 2t \; dt$$

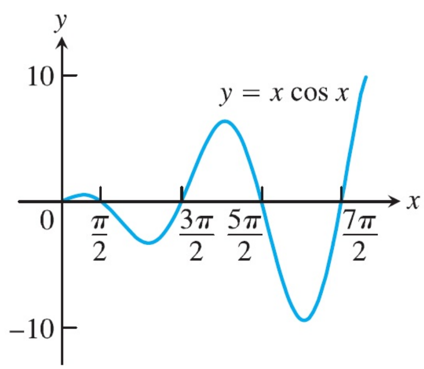
Problem 6

(a) Find the area of the region enclosed by the curve $y = x\cos x$ and the x-axis (see figure) for

(i) $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

(ii) $\frac{3\pi}{2} \leq x \leq \frac{5\pi}{2}$

(iii) $\frac{5\pi}{2} \leq x \leq \frac{7\pi}{2}$



Solution:

Let $I = \int x\cos(x)dx$ be the indefinite integral, then using the integration by parts

$$I = \int x\cos(x)dx = x\sin(x) + \cos(x) + C$$

(i)

$$I\left(\frac{3\pi}{2}\right) - I\left(\frac{\pi}{2}\right) = \frac{3\pi}{2} - \frac{\pi}{2} = 2\pi$$

(ii)

$$I\left(\frac{5\pi}{2}\right) - I\left(\frac{3\pi}{2}\right) = \frac{5\pi}{2} - \frac{3\pi}{2} = 4\pi$$

(iii)

$$I\left(\frac{7\pi}{2}\right) - I\left(\frac{5\pi}{2}\right) = \frac{7\pi}{2} - \frac{5\pi}{2} = 6\pi$$

(b) What pattern do you see, i.e. what is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\pi\right) \leq x \leq \left(\frac{2n+1}{2}\pi\right)$$

where n is an arbitrary positive integer? Give reasons for your answer.

Solution:
$$I\left(\frac{(2n+1)\pi}{2}\right) - I\left(\frac{(2n-1)\pi}{2}\right) = \frac{(2n+1)\pi}{2} - \frac{(2n-1)\pi}{2} = 2n\pi$$

Problem 7

Using the concept of Strategy for Integration "Section 7.5", evaluate the following integrals.

(i)
$$\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$$

Solution:Let $u = \frac{r^3}{18} - 1 \implies du = \frac{r^2}{6}dx$

$$\int r^2\left(\frac{r^3}{18}-1\right)^5 dr = \int 6u^5 du = u^6 + C = \left(\frac{r^3}{18}-1\right)^6 + C$$

(ii)
$$\int \sqrt{\frac{x^4}{x^3-1}}\, dx$$

Solution:Let $u = x^3 - 1 \implies du = 2x^2 dx$

$$\int \sqrt{\frac{x^4}{x^3-1}}\, dx = \int \frac{x^2}{\sqrt{x^3-1}}\, dx = \int \frac{1}{3}\frac{1}{\sqrt{u}} du = \frac{2}{3}\sqrt{u} + C = \frac{2}{3}\sqrt{x^3-1} + C$$

(iii)
$$\int \frac{\sec t \tan t}{\sqrt{\sec t}} dt$$

Solution:Let $u = \sec(t) \implies du = \sec(t)\tan(t)dx$

$$\int \frac{\sec t \tan t}{\sqrt{\sec t}} dt = \int \frac{1}{\sqrt{u}} du = 2\sqrt{u} + C = 2\sqrt{\sec(t)} + C$$

(iv)
$$\int e^x \sin(e^x) dx$$

Solution:Let $u = e^x \implies du = e^x dx$

$$\int e^x \sin(e^x) dx = \int \sin(u) du = -\cos(u) + C = -\cos(e^x) + C$$

(v)

$$\int \sec^6\theta d\theta$$

Solution:

$$\int \sec^6\theta d\theta = \int \sec^4\theta \sec^2\theta d\theta = \int (\tan^2\theta + 1)^2 \sec^2\theta d\theta = \int (\tan^4\theta + 2\tan^2\theta + 1)\sec^2\theta d\theta$$

$$\implies \int \tan^4\theta \sec^2\theta d\theta + 2\int \tan^2\theta \sec^2\theta d\theta + \int \sec^2\theta d\theta$$

$$\implies \frac{1}{5}\tan^5\theta + \frac{2}{3}\tan^3\theta + \tan\theta + C$$

(vi)

$$\int \frac{\sin^3 x}{\cos^4 x} dx$$

Solution:

$$\int \frac{\sin^3 x}{\cos^4 x} dx = \int \frac{\sin^2 x \sin x}{\cos^4 x} dx = \int \frac{(1-\cos^2 x)\sin x}{\cos^4 x} dx = \int \frac{\sin x}{\cos^4 x} dx - \int \frac{\cos^2 x \sin x}{\cos^4 x} dx$$

$$\implies \int \sec^3 x \tan x dx - \int \sec x \tan x dx = \int \sec^2 x \sec x \tan x dx - \int \sec x \tan x dx$$

$$\implies = \frac{1}{3}\sec^3 x - \sec x + C$$

(vii)

$$\int \frac{dx}{4+x^2}$$

Solution:Using the arctan identity, we can write the integral with $a = 2$

$$\int \frac{dx}{4+x^2} = \frac{1}{2}\tan\left(\frac{x}{2}\right) + C$$

(viii)

$$\int \frac{x^3}{x^2-1} dx$$

Solution:Let $u = x^2 - 1 \implies du = 2xdx$

$$\int \frac{x^3}{x^2-1} dx = \int \left(x + \frac{x}{x^2-1}\right) dx = \int x dx + \int \frac{x}{x^2-1} dx = \frac{x^2}{2} + \frac{1}{2}\int \frac{1}{u} du$$

$$\implies \frac{x^2}{2} + \frac{1}{2}\ln|u| + C = \frac{x^2}{2} + \frac{1}{2}\ln|x^2-1| + C$$

(ix)

$$\int \frac{2x^4}{x^3 - x^2 + x - 1} dx$$

Solution:

Since fraction is improper, so using long division and then partial fraction

$$\frac{2x^4}{x^3 - x^2 + x - 1} = 2x + 2 + \frac{2}{x^3 - x^2 + x - 1} = 2x + 2 + \frac{2}{(x^2+1)(x-1)}$$

Factorizing last term using partial fraction

$$\frac{2}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

Using cover method we can find A as

$$\implies A = \frac{2}{1^2+1} = 1$$

By comparing the coefficients we can find the B & C as

$$2 = A(x^2+1) + (Bx+C)(x-1)$$
$$2 = (A+B)x^2 + (-B+C)x + (A-C)$$

by comparing the coefficients

$$x^2: \quad \implies A + B = 0 \tag{1}$$
$$x: \quad \implies C - B = 0 \tag{2}$$
$$\text{constant}: \quad \implies A - C = 2 \tag{3}$$

Solving the above equations $A = 1, B = -1, C = -1$

$$\implies \frac{2x^4}{x^3 - x^2 + x - 1} = 2x + 2 + \frac{1}{x-1} - \frac{x}{x^2+1} - \frac{1}{x^2+1}$$
$$\int \frac{2x^4}{x^3 - x^2 + x - 1} dx = \int \left(2x + 2 + \frac{1}{x-1} - \frac{x}{x^2+1} - \frac{1}{x^2+1}\right) dx$$
$$\int \frac{2x^4}{x^3 - x^2 + x - 1} dx = \int (2x+2)dx + \int \frac{1}{x-1}dx - \int \frac{x}{x^2+1}dx - \int \frac{1}{x^2+1}dx$$
$$= x^2 + 2x + \ln|x-1| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}(x) + C$$

(x)

$$\int \frac{\sin(x)}{\cos^2(x) + \cos(x) - 2} dx$$

Solution:Let $u = \cos(x) \implies du = -\sin(x)dx$

$$\int \frac{\sin(x)}{\cos^2(x) + \cos(x) - 2} dx = \int \frac{-1}{u^2 + u - 2} du = \int \frac{-1}{(u+2)(u-1)} du$$

Using partial fraction

$$\frac{-1}{(u+2)(u-1)} = A\frac{1}{u+2} + B\frac{1}{u-1}$$

Using cover method

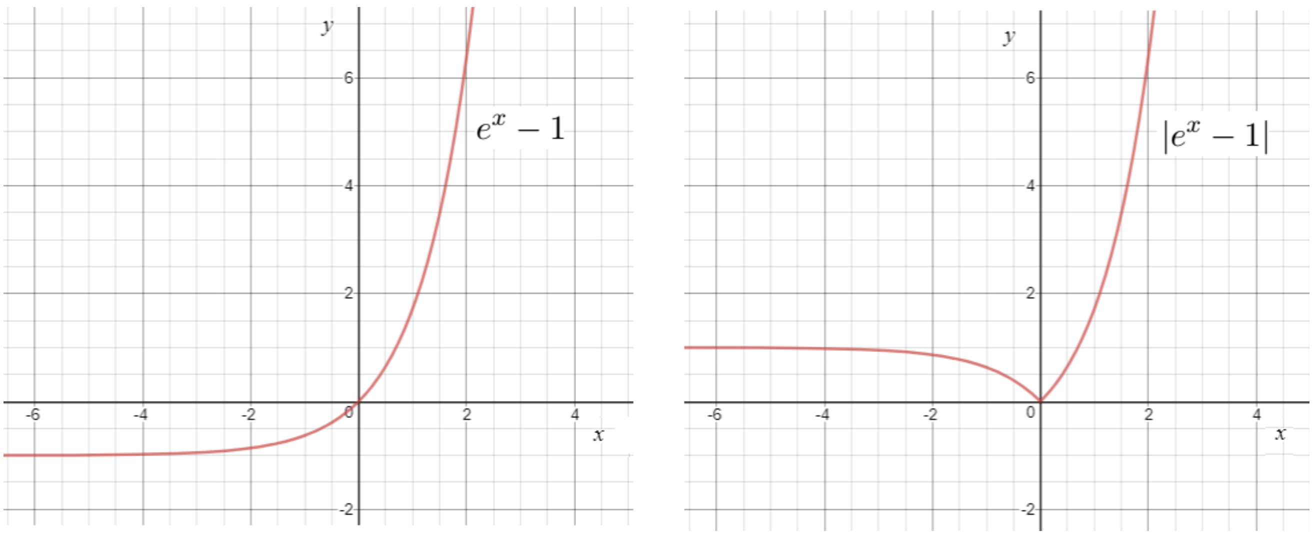
$$A = \frac{1}{3} \quad \text{and} \quad B = -\frac{1}{3}$$

$$\int \frac{-1}{u^2+u-2}du = \frac{1}{3}\int \frac{1}{u+2}du - \frac{1}{3}\int \frac{1}{u-1}du$$
$$\int \frac{-1}{u^2+u-2}du = \frac{1}{3}\ln|(u+2)| - \frac{1}{3}\ln|u-1| + C$$
$$\implies \int \frac{\sin(x)}{\cos^2(x)+\cos(x)-2}dx = \frac{1}{3}\ln\left|\frac{\cos(x)+2}{\cos(x)-1}\right| + C$$

(xi)

$$\int_{-1}^{2} |e^x - 1| dx$$

Solution:



$$|e^x - 1| = \begin{cases} -(e^x-1), & \text{for } x < 0 \\ e^x - 1, & \text{for } x \geq 0 \end{cases}$$

$$\begin{aligned}\int_{-1}^{2} |e^x - 1| dx &= \int_{-1}^{0} -(e^x-1)dx + \int_{0}^{2}(e^x-1)dx \\ &= (x - e^x)|_{-1}^{0} + (e^x - x)|_0^2 \\ &= e^2 + e^{-1} - 3\end{aligned}$$

(xii)

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan(x))}{\sin(x)\cos(x)}dx$$

Let $u = \tan x \implies du = \sec^2 x$ Also if $x = \frac{\pi}{4} \implies u = 1$ and $x = \frac{\pi}{3} \implies u = \sqrt{3}$

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan(x))}{\sin(x)\cos(x)}dx = \int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x}\sec^2 x dx = \int_1^{\sqrt{3}} \frac{\ln(u)}{u}du = \frac{1}{2}\left(\ln(u)\right)^2\Big|_1^{\sqrt{3}} = \frac{1}{2}(\ln(\sqrt{3}))^2$$

Problem 8

Determine whether each integral is convergent or divergent. Evaluate those that are convergent

(i)

$$\int_{-\infty}^{\infty} xe^{-2x^2}dx$$

Solution:

$$\int_{-\infty}^{\infty} xe^{-2x^2}dx = \int_{-\infty}^{0} xe^{-2x^2}dx + \int_{0}^{\infty} xe^{-2x^2}dx$$

$$\implies \int_{-\infty}^{0} xe^{-2x^2}dx = \lim_{t\to-\infty}\left(\frac{-1}{4}\int_{t}^{0} -4xe^{-2x^2}dx\right)$$

$$\int_{-\infty}^{0} xe^{-2x^2}dx = \lim_{t\to-\infty}\frac{-1}{4}e^{-2x^2}\Big|_{t}^{0} = \lim_{t\to-\infty}\frac{-1}{4}(1-e^{-2t^2}) = \frac{-1}{4}$$

$$\implies \int_{0}^{\infty} xe^{-2x^2}dx = \lim_{t\to\infty}\left(\frac{-1}{4}\int_{0}^{t} -4xe^{-2x^2}dx\right)$$

$$\int_{0}^{\infty} xe^{-2x^2}dx = \lim_{t\to\infty}\frac{-1}{4}e^{-2x^2}\Big|_{0}^{t} = \lim_{t\to\infty}\frac{-1}{4}(e^{-2t^2}-1) = \frac{1}{4}$$

$$\implies \int_{-\infty}^{\infty} xe^{-2x^2}dx = \int_{-\infty}^{0} xe^{-2x^2}dx + \int_{0}^{\infty} xe^{-2x^2}dx = \frac{-1}{4}+\frac{1}{4} = 0$$

(ii)

$$\int_{1}^{\infty}\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$$

Solution:Let $u=\sqrt{x} \implies du = \frac{1}{2\sqrt{x}}$, if $x=1 \implies u=1$ and $x=t \implies u=\sqrt{t}$

$$\int_{1}^{\infty}\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{t\to\infty}\int_{1}^{t}\frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{t\to\infty}\int_{1}^{\sqrt{t}} 2e^{-u}du$$

$$= \lim_{t\to\infty} -2e^{-u}|_{1}^{t} = \lim_{t\to\infty} -2e^{-t}+2e^{-1} = 2e^{-1}.$$

(iii)

$$\int_{-\infty}^{\infty}\cos(\pi x)dx$$

Solution:

$$\int_{-\infty}^{\infty}\cos(\pi x)dx = \int_{-\infty}^{0}\cos(\pi x)dx + \int_{0}^{\infty}\cos(\pi x)dx$$

$$\int_{-\infty}^{0}\cos(\pi x)dx = \lim_{t\to-\infty}\int_{t}^{0}\cos(\pi x)dx = \lim_{t\to-\infty}\frac{1}{\pi}\sin(\pi x)\Big|_{t}^{0}$$

$$\implies \int_{-\infty}^{0}\cos(\pi x)dx = \lim_{t\to-\infty} -\frac{1}{\pi}\sin(\pi t) = \text{Limit doesn't converge}$$

$$\int_{0}^{\infty}\cos(\pi x)dx = \lim_{t\to\infty}\int_{0}^{t}\cos(\pi x)dx = \lim_{t\to\infty}\frac{1}{\pi}\sin(\pi x)\Big|_{0}^{t}$$

$$\implies \int_{0}^{\infty}\cos(\pi x)dx = \lim_{t\to\infty}\frac{1}{\pi}\sin(\pi t) = \text{Limit doesn't exits}$$

Hence Actual integral doesn't converge.

(iv)

$$\int_{-\infty}^{\infty} \frac{x^2}{4+x^6} dx$$

Hint: use $t = x^3$

Solution:

$$f(-x) = \frac{(-x)^2}{4+(-x)^6} = \frac{x^2}{4+x^6} = f(x) \qquad \therefore \text{f(x) is even}$$

$$\int_{-\infty}^{\infty} \frac{x^2}{4+x^6} dx = 2\int_{0}^{\infty} \frac{x^2}{4+x^6} dx$$

Let $t = x^3 \implies dt = 3x^2$ if $x = 0$ then $t = 0$ and $x = s$ then $t = s^3$

$$\int_{0}^{\infty} \frac{2x^2}{4+x^6} dx = \lim_{s\to\infty} \int_{0}^{s} \frac{2x^2}{4+x^6} dx = \lim_{s\to\infty} \frac{2}{3} \int_{0}^{s^3} \frac{1}{4+t^2} dt$$

$$= \lim_{s\to\infty} \frac{1}{3} \tan^{-1}\left(\frac{t}{2}\right)\Big|_{0}^{s^3} = \lim_{s\to\infty} \frac{1}{3} \tan^{-1}\left(\frac{s^3}{2}\right) = \frac{\pi}{6}$$

(v)

$$\int_{0}^{\infty} \frac{e^x}{e^{2x}+3} dx$$

Solution:Let $u = e^x \implies du = e^x dx$, if $x = 0$ then $u = 1$ and $x = t$ then $u = e^t$

$$\int_{0}^{\infty} \frac{e^x}{e^{2x}+3} dx = \lim_{t\to\infty} \int_{0}^{t} \frac{e^x}{e^{2x}+3} dx = \lim_{t\to\infty} \int_{1}^{e^t} \frac{1}{u^2+3} du$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Big|_{1}^{e^t}$$

$$= \frac{1}{\sqrt{3}} \lim_{t\to\infty} \left[\tan^{-1}\left(\frac{e^t}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{\sqrt{3}} \left[\frac{\pi}{2} - \frac{\pi}{6}\right] = \frac{\pi\sqrt{3}}{9}$$

(vi)

$$\int_{-1}^{1} \frac{e^{\frac{1}{x}}}{x^3} dx$$

Solution:

$$\int_{-1}^{1} \frac{e^{\frac{1}{x}}}{x^3} dx = \int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3} dx + \int_{0}^{1} \frac{e^{\frac{1}{x}}}{x^3} dx$$

$$\implies \int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^3} dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{e^{\frac{1}{x}}}{x^3} dx$$

Let $u = \frac{1}{x} \implies du = -\frac{1}{x^2} dx$, if $x = -1$ then $u = -1$ and $x = t$ then $u = \frac{1}{t}$

$$\lim_{t\to 0^-} \int_{-1}^{t} \frac{e^{\frac{1}{x}}}{x^3} dx = \lim_{t\to 0^-} \int_{-1}^{1/t} -ue^u du = \lim_{t\to 0^-} (1-u)e^u|_{-1}^{1/t}$$

$$\lim_{t\to 0^-} \left(1-\frac{1}{t}\right) e^{\frac{1}{t}} - \frac{2}{e} = -\frac{2}{e}$$

Now

$$\implies \int_0^1 \frac{e^{\frac{1}{x}}}{x^3}dx = \lim_{t\to 0^+}\int_t^1 \frac{e^{\frac{1}{x}}}{x^3}dx$$

Let $u = \frac{1}{x} \implies du = -\dfrac{1}{x^2}dx$, if $x = 1$ then $u = 1$ and $x = t$ then $u = \dfrac{1}{t}$

$$\lim_{t\to 0^+}\int_t^1 \frac{e^{\frac{1}{x}}}{x^3}dx = \lim_{t\to 0^+}\int_{1/t}^1 -ue^u du = \lim_{t\to 0^+} (1-u)e^u\big|_{1/t}^1$$

$$\lim_{t\to 0^+} -\left(1-\frac{1}{t}\right)e^{\frac{1}{t}} = \text{Limit doesn't exits}$$

Hence overall integral doesn't converge.

(vii)

$$\int_6^8 \frac{4}{(x-6)^3}dx$$

Solution:

$$\int_6^8 \frac{4}{(x-6)^3}dx = \lim_{t\to 6^+}\int_t^8 \frac{4}{(x-6)^3}dx = \lim_{t\to 6^+}\left.\frac{-2}{(x-6)^2}\right|_t^8$$

$$= -2\lim_{t\to 6^+}\left[\frac{1}{4}-\frac{1}{(t-6)^2}\right] = \text{limit diverge so does integral}$$
