MT110: Calculus and Analytic Geometry

Homework 5 Solution

Due: Fri, Dec 7 *Fall 2018*

Problem 1

Evaluate the following limits.

(a) $\lim_{x \to -2} \frac{x+2}{\ln(x+3)}$

Which is an indeterminant form of $\frac{0}{0}$
By l'Hopital's Rule,

$$\lim_{x \to -2} \frac{1}{\frac{1}{x+3}} = \lim_{x \to -2} x + 3 = 1$$

(b) $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$

Which is an indeterminant form of $\frac{\infty}{\infty}$
By l'Hopital's Rule,

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0$$

(c) $\lim_{x \to \infty} x \tan \frac{1}{x}$
Which is an indeterminant form of $(\infty)(0)$
Converting to $\frac{0}{0}$:

$$\lim_{x \to \infty} \frac{\tan \frac{1}{x}}{\frac{1}{x}}$$

By l'Hopital's Rule,

$$= \lim_{x \to \infty} \frac{(\sec^2 \frac{1}{x}) \cdot \left(\frac{-1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x \to \infty} \sec^2 \frac{1}{x} = \sec^2(0) = 1$$

(d) $\lim_{x \to 0^+} x^{-\sqrt{x}}$
let

$$y = \lim_{x \to 0^+} x^{-\sqrt{x}}$$

Which is an indeterminant form of 0^0

$$\ln y = \ln \lim_{x \to 0^+} x^{-\sqrt{x}} = \lim_{x \to 0^+} -\sqrt{x} \ln x$$

It is an indeterminant form of $(0)(\infty)$

$$\ln y = \lim_{x \to 0^+} -\frac{\ln x}{\frac{1}{\sqrt{x}}}$$

Which is an indeterminant form of $\dfrac{\infty}{\infty}$

By l'Hopital's Rule,

$$\ln y = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{1}{2x^{\frac{3}{2}}}}$$

$$\ln y = \lim_{x \to 0^+} \frac{2x^{\frac{3}{2}}}{x}$$

$$\ln y = \lim_{x \to 0^+} 2\sqrt{x} = 0$$

$$e^{\ln y} = e^0$$

$$y = 1$$

(e) $\displaystyle\lim_{x \to 0^+} \frac{\ln x}{x}$

As

$\ln x \to -\infty$

$x \to 0^+$

$\implies \dfrac{\ln x}{x} \to -\infty$

$$\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$$

(f) $\displaystyle\lim_{x \to 1^+} x^{\frac{2}{1-x}}$

let

$$y = \lim_{x \to 1^+} x^{\frac{2}{1-x}}$$

$$\ln y = \ln \lim_{x \to 1^+} x^{\frac{2}{1-x}}$$

$$\ln y = \lim_{x \to 1^+} \frac{2\ln x}{1-x}$$

By l'Hopital's Rule,

$$\ln y = \lim_{x \to 1^+} \frac{2\frac{1}{x}}{-1}$$

$$\ln y = -2$$

$$e^{\ln y} = e^{-2}$$

$$y = e^{-2}$$

Problem 2

Read the curve sketching guidelines on page 311 in section 4.5 (Stewart) to sketch the following curves by hand. Mention all the steps. Note: Reading the examples in that section will be very helpful.

(a) $y = x(x-4)^3$

(A) Start by defining the Domain of function: $x \in (-\infty, \infty)$

(B) Then set $y = 0$ and solve it to find the points when curve inersects the x axis.
$y = x(x-4)^3 = 0$
$x = 0, x = 4$

(C) Now checking for symmetry:

(i) $f(-x) = f(x)$
$f(-x) = -x(-x-4)^3 \neq f(x)$

(ii) $f(-x) = -f(x)$
$f(-x) = -x(-x-4)^3 \neq -f(x)$

(iii) $f(x+p) = -f(x)$
$f(x+p) = (x+p)(x+p-4)^3 \neq f(x)$ for any positive p.

So the given function doesn't have any kind of symmetry.

(D) Now checkig for asymptote: $\lim_{x\to-\infty} x(x-4)^3 = \infty$ $\lim_{x\to\infty} x(x-4)^3 = \infty$

(E) Intervals of Increase or Decrease:
Find $f'(x) = 4(x-4)^2(x-1) = 0$
$x = 1,\ x = 4$

interval	$f'(x)$	Increasing or decreasing
$(-\infty, 1)$	-ive	decreasing
$(1, 4)$	+ive	increasing
$(4, \infty)$	+ive	increasing

(F) Critical points of $f(x)$ would be: $f'(c) = 0$
$x = 1,\ x = 4$ For $x = 1$ $f'(x)$ changes from negative to positive so, it is a local minima.

(G) Concavity and Points of Inflection: $f''(x) = 12(x^2 - 6x + 8) = 0$ $x = 2,\ x = 4$

interval	$f''(x)$	Concavity
$(-\infty, 2)$	+ive	Concave Upward
$(2, 4)$	-ive	Concave Downward
$(4, \infty)$	+ive	Concave Upward

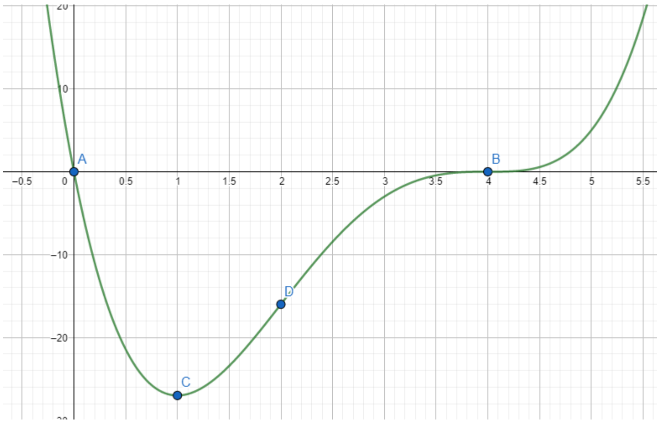
Point of Inflection $x = 2,\ x = 4$

(H) Sketch the Curve: Using the information in items A–G, draw the graph.

(i) Sketch the asymptotes as dashed lines.

(ii) Plot the intercepts, maximum and minimum points, and inflection points.

(iii) Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.



(b) $y = (1-x)e^x$

(A) Start by defining the Domain of function: $x \in (-\infty, \infty)$

(B) Then set $y = 0$ and solve it to find the points when curve inersects the x axis.
$y = (1-x)e^x = 0$
$x = 1$ if $y = 1$ then $x = 0$.

(C) Now checking for symmetry:

(i) $f(-x) = f(x)$
$f(-x) = (1+x)e^{-x} \neq f(x)$

(ii) $f(-x) = -f(x)$
$f(-x) = (1+x)e^{-x} \neq -f(x)$

(iii) $f(x+p) = -f(x)$
$f(x+p) = (1-x-p)e^{x+p} \neq f(x)$ for any positive p.

So the given function doesn't have any kind of symmetry.

(D) Now checkig for asymptote: $\lim_{x\to-\infty}(1-x)e^x = 0$ $\lim_{x\to\infty}(1-x)e^x = \infty$
Similarly vertical asymptotes: $\lim_{x\to 4}(1-x)e^x = 3e^4$ Which is rapidly increasing as x is increasing.

(E) Intervals of Increase or Decrease:
Find $f'(x) = -xe^x = 0,\ \ x = 0$

interval	$f'(x)$	Increasing or decreasing
$(-\infty, 0)$	+ive	increasing
$(0, \infty)$	-ive	decreasing

(F) Critical points of $f(x)$ would be: $f'(c) = 0$
$x = 0$
For $x = 0$ $f'(x)$ changes from positive to negative so, it is a local maxima.

(G) Concavity and Points of Inflection:
$f''(x) = -e^x(x+1) = 0$

interval	$f''(x)$	Concavity
$(-\infty, -1)$	+ive	Concave Upward
$(-1, \infty)$	-ive	Concave Downward

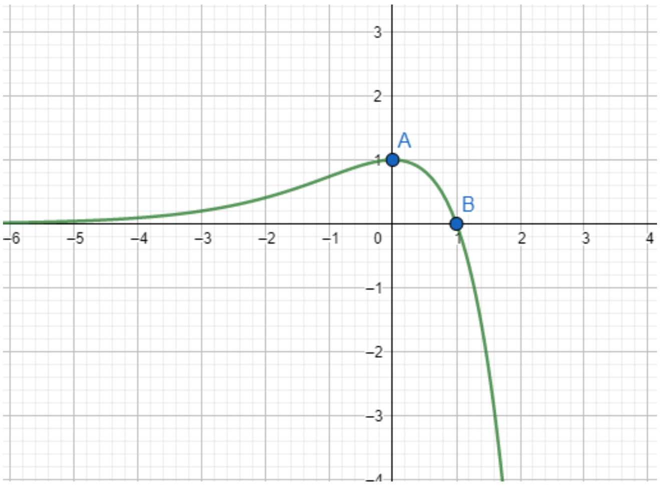
Point of Inflection $x = -1$

(H) Sketch the Curve: Using the information in items A–G, draw the graph.

(i) Sketch the asymptotes as dashed lines.

(ii) Plot the intercepts, maximum and minimum points, and inflection points.

(iii) Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.



(c) $y = \dfrac{x}{x^3 - 1}$

(A) Start by defining the Domain of function: $x \in (-\infty, \infty) - 1$

(B) Then set $y = 0$ and solve it to find the points when curve inersects the x axis.
$y = \dfrac{x}{x^3 - 1} = 0$
$x = 0$

(C) Now checking for symmetry:

(i) $f(-x) = f(x)$
$f(-x) = \dfrac{-x}{-x^3 - 1} \neq f(x)$

(ii) $f(-x) = -f(x)$
$f(-x) = \dfrac{-x}{-x^3 - 1} \neq -f(x)$

(iii) $f(x + p) = -f(x)$
$f(x + p) = \dfrac{x + p}{(x + p)^3 - 1} \neq f(x)$ for any positive p.

So the given function doesn't have any kind of symmetry.

(D) Now checkig for asymptote: $\lim_{x \to -\infty} \dfrac{x}{x^3 - 1} = 0$
$\lim_{x \to \infty} \dfrac{x}{x^3 - 1} = 0$
Similarly vertical asymptotes: $\lim_{x \to 1^+} \dfrac{x}{x^3 - 1} = \infty$
$\lim_{x \to 1^-} \dfrac{x}{x^3 - 1} = -\infty$

(E) Intervals of Increase or Decrease: Find $f'(x) = \dfrac{-2x^3 - 1}{(x^3 - 1)^2} = 0, \;\; x = -0.794$

interval	$f'(x)$	Increasing or decreasing
$(-\infty, -0.794)$	+ive	increasing
$(-0.794, \infty)$	-ive	decreasing

(F) Critical points of $f(x)$ would be: $f'(c) = 0$
$x = -0.794$
For $x = -0.794$ $f'(x)$ changes from positive to negative so, it is a local maxima.

(G) Concavity and Points of Inflection:
$f''(x) = \dfrac{6x^2(x^3 + 2)}{(x^3 - 1)^3}$

interval	$f''(x)$	Concavity
$(-\infty, -1.26)$	+ive	Concave Upward
$(-1.26, 0)$	-ive	Concave Downward
$(0, \infty)$	-ive	Concave Downward

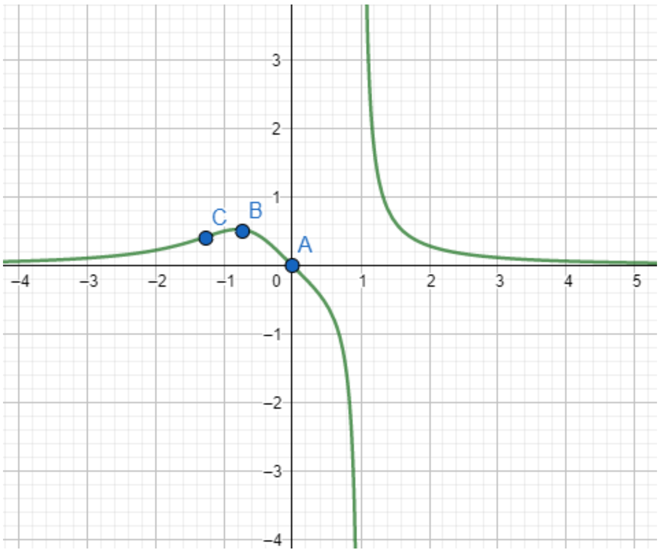
Point of Inflection $x = -1.26$

(H) Sketch the Curve: Using the information in items A–G, draw the graph.

(i) Sketch the asymptotes as dashed lines.

(ii) Plot the intercepts, maximum and minimum points, and inflection points.

(iii) Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.



(d) $y = \dfrac{\sin x}{2 + \cos x}$

(A) Start by defining the Domain of function: $x \in (-\infty, \infty)$

(B) Then set $y = 0$ and solve it to find the points when curve inersects the x axis.

$y = \dfrac{\sin x}{2 + \cos x} = 0$

$x = 0, n\pi$

(C) Now checking for symmetry:

(i) $f(-x) = f(x)$

$f(-x) = \dfrac{-\sin x}{2 + \cos x} \neq f(x)$ Not Even function.

(ii) $f(-x) = -f(x)$

$f(-x) = \dfrac{-\sin x}{2 + \cos x} = -f(x)$ So, the function is Odd function.

(iii) $f(x + p) = -f(x)$

$f(x + 2\pi) = \dfrac{\sin(x + 2\pi)}{2 + \cos(x + 2\pi)} = f(x)$

Function is periodic with a period of 2π.

(D) Now checkig for asymptote: $\lim_{x \to -\infty} \dfrac{x}{x^3 - 1} = 0$

$\lim_{x \to \infty} \dfrac{x}{x^3 - 1} = 0$

Similarly vertical asymptotes: $\lim_{x \to 1^+} \dfrac{x}{x^3 - 1} = \infty$

$\lim_{x \to 1^-} \dfrac{x}{x^3 - 1} = -\infty$

(E) Intervals of Increase or Decrease: Find $f'(x) = \dfrac{\sin^2 x + \cos^2 x + \cos x}{(1 + \cos x)^2} = 0$,
$x = 2.2,\ x = -2.2$

interval	$f'(x)$	Increasing or decreasing
$(-\infty, -2.2)$	-ive	deccreasing
$(-2.2, 2.2)$	+ive	increasing
$(2.2, \infty)$	-ive	decreasing

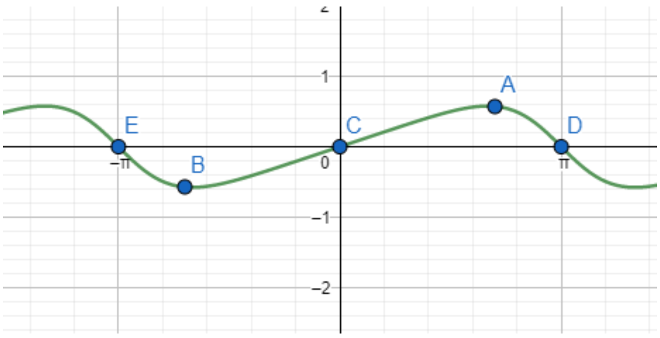
(F) Critical points of $f(x)$ would be: $f'(c) = 0$
No critical Points.

(G) Concavity and Points of Inflection:
$f''(x) = \dfrac{\sin x(2\sin^2 x + 2\cos^2 x + \cos x - 1)}{1 + \cos x} = 0$ roots: $x = 2n\pi$

interval	$f''(x)$	Concavity
$(-\pi, 0)$	-ive	Concave Downward
$(0, \pi)$	+ive	Concave Upward
$(\pi, 2\pi)$	-ive	Concave Downward
$(2\pi, 3\pi)$	+ive	Concave Upward

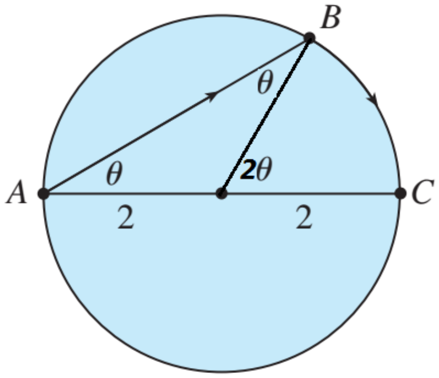
Point of Inflection $x = 0, 2n\pi$

(H) Sketch the Curve: Using the information in items A–G, draw the graph.
  (i) Sketch the asymptotes as dashed lines.
  (ii) Plot the intercepts, maximum and minimum points, and inflection points.
  (iii) Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.



Problem 3

A woman at a point A on the shore of a circular lake with radius 2 km wants to arrive at the point C diametrically opposite to A on the other side of the lake. She can walk at the rate of 4 km/h and row a boat at 2 km/h. Find the path that she must take in order to go from A to C in the shortest possible time?



Distance travelled by rowing: line AB
Let $\angle AOB = 180 - 2\theta$
Then by sine rule

$$\frac{AB}{\sin(\pi - 2\theta)} = \frac{2}{\sin\theta}$$
$$AB = \frac{2\sin(\pi - 2\theta)}{\sin\theta}$$
$$AB = d_r = 4\cos\theta$$

Time taken to travel this distance is:

$$t_r = \frac{d_r}{v_r} = \frac{4\cos\theta}{2} = 2\cos\theta$$

Similarly the distance travelled through walking is: arc BC

$$d_w = (2)(2)(\theta) = 4\theta$$

Time taken to travel this distance is:

$$t_w = \frac{d_w}{v_w} = \frac{4\theta}{4} = \theta$$

Total distance travelled:

$$T(\theta) = t_r + t_w = 2\cos\theta + \theta$$
$$T'(\theta) = -2\sin\theta + 1 = 0 \Longrightarrow \theta = \frac{\pi}{6}$$

The function is defined on the given interval $\theta \in [0, \frac{\pi}{2}]$ so by applying closed interval method.

1) If she walks all the way to point C. $\theta = \frac{\pi}{2}$

$$T\left(\frac{\pi}{2}\right) = 1.57$$

2) If she rows all the way to point C.$\theta = 0$

$$T(0) = 2$$

3) At the criticle point $\theta = \frac{\pi}{6}$

$$T\left(\frac{\pi}{6}\right) = 2.25$$

So, The minimum value occurs at $\theta = \frac{\pi}{2}$ which is why the best way will be for woman to walk all the way.

Problem 4

A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other piece is bent into a circle. What is the length of each piece so that the total area enclosed in the two shapes is

(a) a maximum?

(b) a minimum?

(c) Evaluate the maximum and minimum possible enclosed areas.

Let x = length of the portion that is bent into a circle
Then, 10-x = length bent into a square
Each side of the square has length $\frac{10-x}{4}$
So, area of square = $\frac{(10-x)^2}{16}$
x = circumference of circle = $2\pi r$ $\quad So, \ r = \frac{x}{2\pi}$
Therefore, Area of circle = $\pi r^2 = \frac{x^2}{4\pi}$

$$T = total\ area = \frac{(10-x)^2\pi + 4x^2}{16\pi}$$
$$= \frac{(100-20x+x^2)\pi + 4x^2}{16\pi}$$
$$= \frac{(4+\pi)x^2 - 20\pi x + 100\pi}{16\pi}$$
$$T = \frac{(4+\pi)x^2}{16\pi} - \frac{5}{4}x + \frac{25}{4} \quad for\ x \in [0, 10]$$

The graph of T is a parabola opening upward. The minimum of T occurs at the vertex of the parabola.

$$T'(x) = \frac{(4+\pi)x}{8\pi} - \frac{5}{4} = 0 \Longrightarrow x = \frac{10\pi}{4+\pi}$$

(a) a maximum?

Find the values of T at $x = 0$, $x = 10$ and $x = \frac{10\pi}{4+\pi}$.

So, $T(0) = \frac{25}{4}$ and

$$T(10) = \frac{(4+\pi)100}{16\pi} - \frac{50}{4} + \frac{25}{4} = \frac{400 + 100\pi - 200\pi + 100\pi}{16\pi} = \frac{400}{16\pi} = \frac{100}{4\pi}$$
$$T\left(\frac{10\pi}{4+\pi}\right) = \frac{(4+\pi)100\pi^2}{(16\pi)(4+\pi)^2} - \frac{(5)(10\pi)}{4(4+\pi)} + \frac{25}{4}$$

$$T\left(\frac{10\pi}{4+\pi}\right) = \frac{25\pi - 50\pi + 25(4+\pi)}{4(4+\pi)} = \frac{25\pi - 50\pi + 100 + 25\pi}{4(4+\pi)} = \frac{100}{4(4+\pi)}$$

$$T\left(\frac{10\pi}{4+\pi}\right) = \frac{25}{4+\pi}$$

So, maximum area is at x=10.

(b) a minimum?

It has its minimum value at $x = \frac{10\pi}{4+\pi}$.

$$T\left(\frac{10\pi}{4+\pi}\right) = \frac{25}{4+\pi}$$

(c) Evaluate the maximum and minimum possible enclosed areas.

For minimum area:

Area of Circle= $\dfrac{\left(\frac{10\pi}{4+\pi}\right)^2}{4\pi} = \dfrac{100\pi^2}{4\pi(4+\pi)^2}$

Area of the square= $\dfrac{\left(10 - \frac{10\pi}{4+\pi}\right)^2}{16} = \dfrac{(40 + 10\pi - 10\pi)^2}{16(4+\pi)^2} = \dfrac{100}{(4+\pi)^2}$

Total Area= $\dfrac{25}{4+\pi}$

For maximum area:

Area of Circle= $\dfrac{(10)^2}{4\pi} = \dfrac{100}{4\pi}$

Area of the square= 0

Total Area= $\dfrac{100}{4\pi}$

Problem 5

For a fish swimming at a speed v relative to the water, the energy expenditure per unit time is proportional to v^3. It is believed that migrating fish try to minimize the total energy required to swim a fixed distance. If the fish are swimming against an ocean current u (for $u < v$), then the time required to swim a distance L is $\frac{L}{v-u}$ and the total energy E required to swim the distance is given by

$$E(v) = av^3 \frac{L}{v-u},$$

where a is the constant of proportionality.

(a) Determine the value of v that minimizes E.

$$\frac{dE}{dv} = L\frac{(v-u)3av^2 - av^3}{(v-u)^2} = 0$$

$$L\frac{3av^3 - 3av^2u - av^3}{(v-u)^2} = 0$$

$$L(2av^3 - 3av^2u) = 0$$

As $L \neq 0$
So,

$$2av^3 - 3av^2u = 0$$

$$2av^2(2v - 3u) = 0$$

$E(v)$ will have its minimum value at $v = \dfrac{3u}{2}$

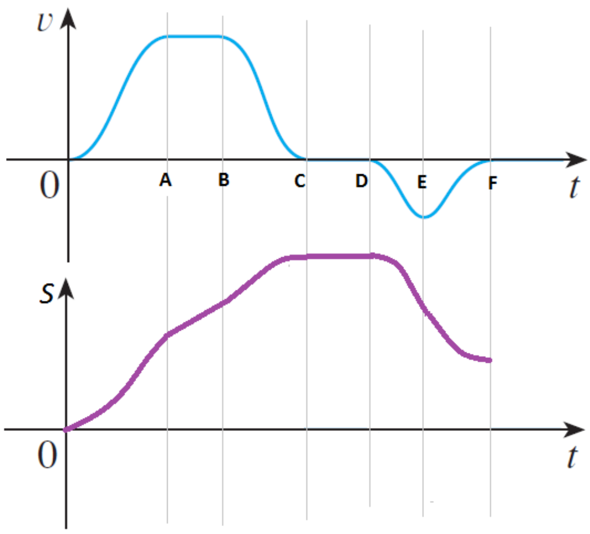
(b) Sketch the graph of E.
let,$a = 2.$ and $u = 2.$

(c) It has been experimentally found that the migrating fish swim against an ocean current at a speed that is 50% greater than the current speed. How accurate is the model equation $E(v)$?
If we consider E minimum at $v > 1.5u$ then in this case E also gives a minimum value at $v = 1.5u$. So, the given model is quite acurate.

Problems 6

The graph of the velocity function of a particle is shown in the figure. Sketch the graph of a position function.

interval	$V(t)$	S(t)
$(0, A)$	+ive	Increasing
(A, B)	+ive	Increasing
(B, C)	+ive	Increasing
(C, D)	zero	Constant
(D, E)	-ive	decreasing
(E, F)	-ive	decreasing



Problems 7

A particle is moving with the given data. Find the position of the particle.

$$a(t) = 3\cos t - 2\sin t,\ \ s(0) = 0,\ \ v(0) = 4$$

Take the antiderivative of $a(t)$ which will result in $v(t)$

$$v(t) = 3\sin t + 2\cos t + c_1$$

Putting $v(0) = 4$

$$v(0) = 3\sin(0) + 2\cos(0) + c_1 = 4$$
$$c_1 = 2$$
$$v(t) = 3\sin t + 2\cos t + 2$$

Now taking the antiderivative of $v(t)$ we will get $s(t)$

$$s(t) = -3\cos t + 2\sin t + 2t + c_2$$

Putting $s(0) = 0$

$$s(0) = -3\cos(0) + 2\sin(0) + 2(0) + c_2 = 0$$
$$c_2 = 3$$

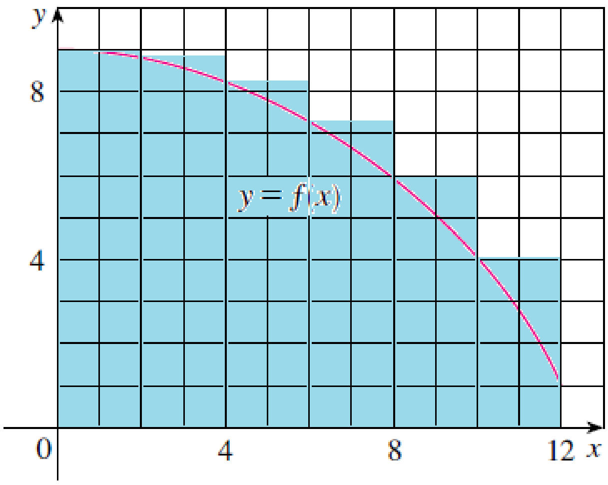
Which wil result in:

$$s(t) = -3\cos t + 2\sin t + 2t + 3$$

Problems 8

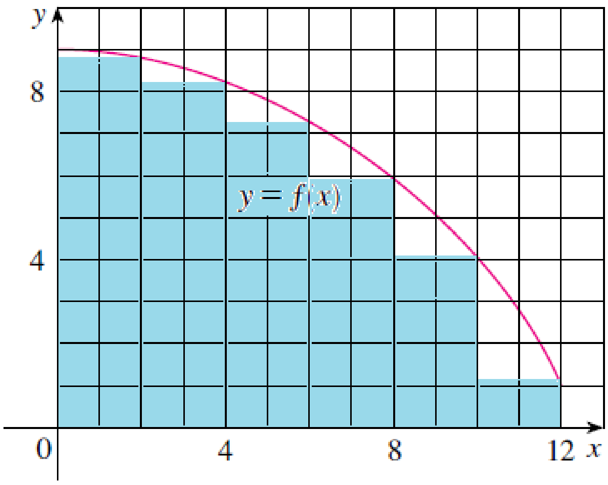
(a) Use six rectangles to find estimates of each type for the area under the given graph of f from $x = 0$ to $x = 12$.

(i) L_6 (sample points are left endpoints)
Area$= (9)(2) + (8.9)(2) + (8.2)(2) + (7.2)(2) + (6)(2) + (4)(2) = 86.6$



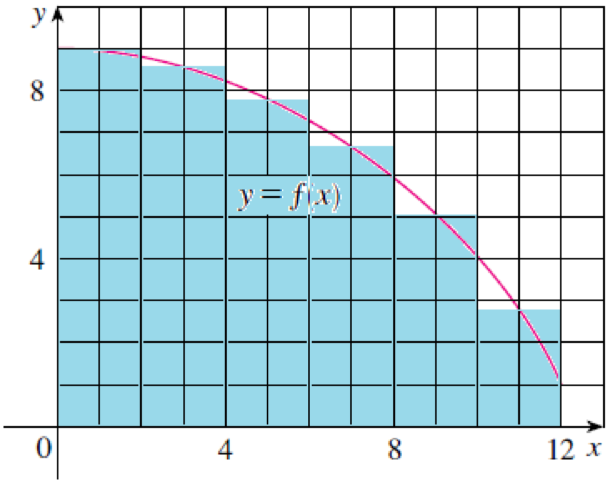
x_i	$f(x_i)$	$f(x_i)\Delta x$	$f(x_{i-1})$	$f(x_{i-1})\Delta x$
0	9	18	-	-
2	8.9	17.8	8.9	17.8
4	8.2	16.4	8.2	16.4
6	7.2	14.4	7.2	14.4
8	6	12	6	12
10	4	8	4	8
12	-	-	1.2	2.4
-	-	$L_6 = 71.2$	-	$R_6 = 86.6$

(ii) R_6 (sample points are right endpoints)
Area$= (8.9)(2) + (8.2)(2) + (7.2)(2) + (6)(2) + (4.1)(2) + (1.2)(2) = 71.2$



(iii) M_6 (sample points are midpoints)

Area$= (9)(2) + (8.5)(2) + (7.8)(2) + (6.7)(2) + (5)(2) + (2.8)(2) = 79.6$



x_i	$f(x_i)$	$f(x_i)\Delta x$
1	9	18
3	8.5	17
5	7.8	15.6
7	6.7	13.4
9	5	10
11	2.8	5.6
-	-	$M_6 = 79.6$

(b) Is L_6 an underestimate or overestimate of the true area?
Overesimate of the true area.

(c) Is R_6 an underestimate or overestimate of the true area?
Underestimate of the true area.

(d) Which of the numbers L_6, R_6, M_6 or gives the best estimate? Explain.
M_6 gives the best estimate of the Area, because it takes rectangles based on the mid point which is neither much of an oerestimate of the true area nor an much of an underestimate of the true area.

Problem 9

Speedometer readings for a motorcycle at 12-second intervals are given in the table.

$t(s)$	0	12	24	36	48	60
$v(ft/s)$	30	28	25	22	24	27

(a) Estimate the distance traveled by the motorcycle during this time period using the velocities at the beginning of the time intervals.

x_i	$f(x_i)$	$f(x_i)\Delta x$	$f(x_{i-1})$	$f(x_{i-1})\Delta x$
0	30	360	-	-
12	28	336	28	336
24	25	300	25	300
36	22	264	22	264
48	24	288	24	288
60	-	-	27	324
-	-	$L_5 = 1548\ ft$	-	$R_5 = 1512\ ft$

Distance$= (30)(12) + (28)(12) + (25)(12) + (22)(12) + (24)(12) = 1548\ ft$

(b) Give another estimate using the velocities at the end of the time periods.
Distance$= (28)(12) + (25)(12) + (22)(12) + (24)(12) + (27)(12) = 1512\ ft$

(c) Are your estimates in parts (a) and (b) upper and lower estimates? Explain.

The estimate in part a is an upperestimate as speed is decreasing throughout these intervals and by assuming the initial value is constant we are overestimating the distance.

Problem 10

Verify that R_n and L_n gives the same value as $\lim_{n\to\infty}$ for the given integral

$$\int_1^4 (x^2 - 4x + 2)\,\mathrm{d}x.$$

Computing

$$\int_1^4 (x^2 - 4x + 2)\,\mathrm{d}x = \frac{x^3}{3} - \frac{4x^2}{2} + 2x = -3$$

Now, for R_n, $\Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}$

$$R_n = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(1 + \frac{3i}{n}\right)^2 - 4\left(1 + \frac{3i}{n}\right) + 2 \right]$$

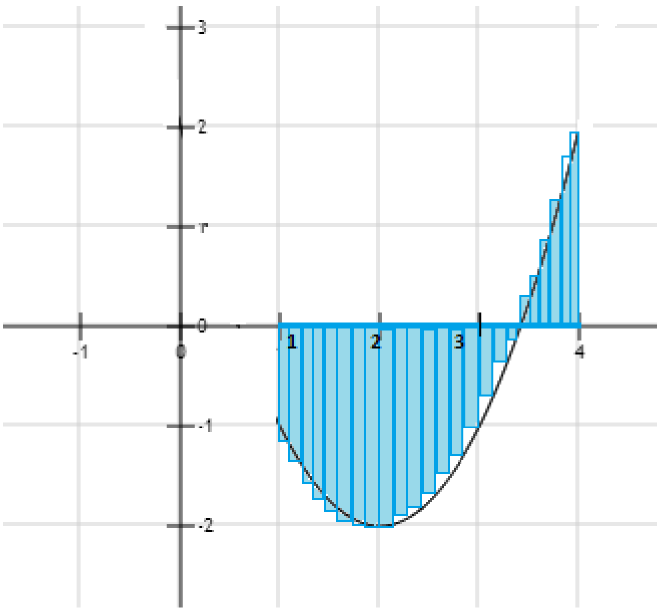
$$R_n = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{9i^2}{n^2} - \frac{6i}{n} - 1 \right]$$

$$R_n = \lim_{n\to\infty} \frac{3}{n} \left[\frac{9\sum_{i=1}^{n} i^2}{n^2} - \frac{6\sum_{i=1}^{n} i}{n} - \sum_{i=1}^{n} 1 \right]$$

$$R_n = \lim_{n\to\infty} \frac{3}{n} \left[\frac{9}{n^2} \frac{n(n+1)(2n+1)}{6} - \frac{6}{n} \frac{n(n+1)}{2} - n \right]$$

$$R_n = \lim_{n\to\infty} \left[\frac{9}{2} \left(\frac{2n^3 + 3n^2 + n}{n^3} \right) - 9\left(\frac{n^2 + n}{n^2} \right) - \frac{3n}{n} \right]$$

$$R_n = \lim_{n\to\infty} \left[\frac{9}{2} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) - 9\left(\frac{1}{n} + 1 \right) - 3 \right] = -3$$



Similarly,

$$L_n = \lim_{n\to\infty} \frac{3}{n} \sum_{i=0}^{n-1} \left[\left(1 + \frac{3i}{n}\right)^2 - 4\left(1 + \frac{3i}{n}\right) + 2 \right]$$

Let $k = n - 1$,

$$L_n = \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left[\frac{9k^2}{n^2} - \frac{6k}{n} - 1 \right] - 1$$

$$L_n = \lim_{n\to\infty} \frac{3}{n} \left[\frac{9\sum_{k=1}^{n} k^2}{n^2} - \frac{6\sum_{k=1}^{n} k}{n} - \sum_{k=1}^{n} 1 \right] - 1$$

$$L_n = \lim_{n\to\infty} \frac{3}{n} \left[\frac{9}{n^2} \frac{k(k+1)(2k+1)}{6} - \frac{6}{n} \frac{k(k+1)}{2} - k \right]$$

$$L_n = \lim_{n\to\infty} \frac{3}{n} \left[\frac{9}{n^2} \frac{n(n-1)(2n-1)}{6} - \frac{6}{n} \frac{n(n+1)}{2} - n + 1 \right]$$

$$L_n = \lim_{n\to\infty} \left[\frac{9}{2} \left(\frac{2n^3 - 3n^2 + n}{n^3} \right) - 9 \left(\frac{n^2 + n}{n^2} \right) - \frac{3n}{n} \right]$$

$$L_n = \lim_{n\to\infty} \frac{9}{2} \left(2 + \frac{3}{n} + \frac{1}{n^2} \right) - 9 \left(\frac{1}{n} + 1 \right) - 3 = -3$$