MT110: Calculus and Analytic Geometry

Homework 4 Solution

Due: Fri, Nov 23, 2:00 pm *Fall 2018*

Problem 1

Find the derivative of the following functions using chain rule.

(a) $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$

Solution: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ where $u = \frac{x^2}{8} + x - \frac{1}{x}$

$$\frac{dy}{dx} = 4\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3 \left(\frac{x}{4} + 1 + \frac{1}{x^2}\right)$$

(b) $y = \cot\left(\pi - \frac{1}{x}\right)$

Solution: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ where $u = \pi - \frac{1}{x}$

$$\frac{dy}{dx} = -\csc^2\left(\pi - \frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = -\csc^2\left(\pi - \frac{1}{x}\right)\left(\frac{1}{x^2}\right)$$

(c) $y = (\csc(x) + \cot(x))^{-1}$

Solution: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ where $u = \csc(x) + \cot(x)$

$$\frac{dy}{dx} = \frac{-1}{(\csc(x) + \cot(x))^2}\left(-\csc(x)\cot(x) - \csc^2(x)\right) = \frac{\left(\csc(x)\cot(x) + \csc^2(x)\right)}{(\csc(x) + \cot(x))^2}$$

(d) $y = 4\sin\left(\sqrt{1 + \sqrt{t}}\right)$

Solution: $\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dt}$ where $u = \sqrt{1 + v}, \quad v = \sqrt{t}$

$$\frac{dy}{dt} = 4\cos\left(\sqrt{1 + \sqrt{t}}\right)\left(\frac{1}{2\sqrt{1 + \sqrt{t}}}\right)\left(\frac{1}{2\sqrt{t}}\right) = \frac{\cos\left(\sqrt{1 + \sqrt{t}}\right)}{\sqrt{t}\left(\sqrt{1 + \sqrt{t}}\right)}$$

Problem 2

Find the derivative of the following functions using implicit differentiation or use the logarithmic differentiation where applicable.

(a) $y \sin\left(\frac{1}{y}\right) = 1 - xy$

Solution: diff w.r.t x and $\frac{dy}{dx} = y'$

$$y' \sin\left(\frac{1}{y}\right) + y \cos\left(\frac{1}{y}\right)\left(\frac{-1}{y^2}\right) y' = -y - xy'$$

$$\implies \left[\sin\left(\frac{1}{y}\right) - \frac{\cos\left(\frac{1}{y}\right)}{y} + x\right] y' = -y$$

$$\implies y' = \frac{-y}{\sin\left(\frac{1}{y}\right) - \frac{1}{y}\cos\left(\frac{1}{y}\right) + x}$$

(b) $y^2 \cos\left(\frac{1}{y}\right) = 2x + 2y$

Solution: diff w.r.t x and $\frac{dy}{dx} = y'$

$$2y \cos\left(\frac{1}{y}\right) y' + y^2 \sin\left(\frac{1}{y}\right)\left(\frac{1}{y^2}\right) y' = 2 + 2y'$$

$$\implies y' = \frac{2}{2y\cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) - 2}$$

(c) $x^{\ln(x)}$

Solution: $y = x^{\ln(x)}$ taking log on both sides

$\ln y = \ln(x^{\ln(x)})$

$\ln y = \ln(x)\ln(x) \implies \ln y = (\ln(x))^2$

Diff w.r.t x

$$\frac{1}{y} y' = 2\ln(x)\left(\frac{1}{x}\right)$$

$$\implies y' = 2x^{\ln(x)-1}\ln(x)$$

(d) $x^y = y^x$

Solution: Taking log on both sides

$y \ln x = x \ln(y)$

Diff w.r.t x

$$y' \ln(x) + \frac{y}{x} = \ln(y) + \frac{x}{y} y'$$

$$\implies y' = \frac{\ln(y) - y/x}{\ln(x) - x/y}$$

Problem 3

Find the equation of the line tangent to the curve $xy - 2x - y = 0$ at $x = 4$

Solution: The gradient can be found by differentiating implicitly when $x = 4$ and

$4y - 8 - y = 0 \implies y = \frac{8}{3}$, so point becomes $P = (4, 8/3)$

$xy - 2x - y = 0$ Diff w.r.t x

$$y + xy' - 2 - y' = 0 \implies y' = \frac{2-y}{x-1}\bigg|_{(4,8/3)} = -\frac{2}{9}$$

The equation of tangent line is

$$y - \frac{8}{3} = -\frac{2}{9}(x-4) \implies 2x + 9y = 32$$

Problem 4

Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min as shown in Fig. 1. It forms a pile in the shape of a right circular cone whose base diameter and height are always the same. How fast is the height of the pile increasing when the pile is 17 ft high?

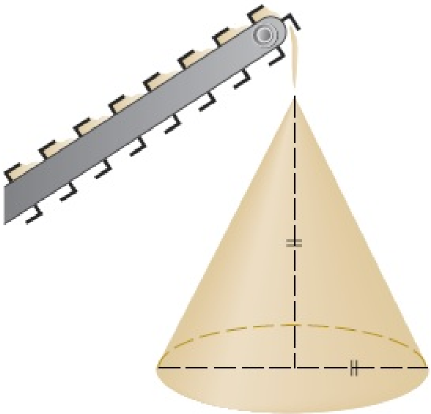


Figure 1: Pile of dump

Solution:
Since height h and diameter is always same then $h = 2r \implies r = h/2$. Let V be the volume of cone then $V = \frac{1}{3}\pi r^2 h$

$\implies V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$

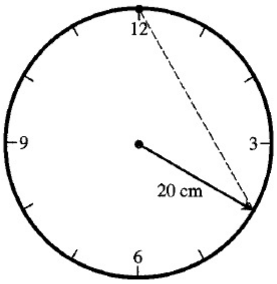
Diff w.r.t t

$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \quad \because \frac{dV}{dt} = 30ft^3/min$

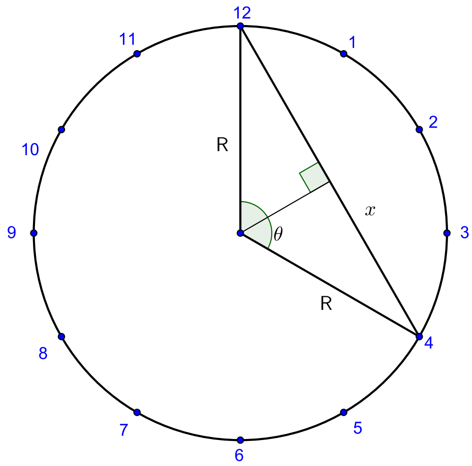
$\implies \frac{dh}{dt} = \frac{dV}{dt}\frac{4}{\pi}\frac{1}{h^2} = \frac{120}{289\pi} = 0.132ft/min$

Problem 5

At what rate is the distance between the tip of second hand and the 12 o'clock mark changing when the second hand points to 4 o'clock?



Solution:



Since inside clock there is isosceles triangle so we can draw a perpendicular from origin onto x which will bisect the angle and length of segment x. Now using the any right angle triangle, we can write this

$$\sin\left(\frac{\theta}{2}\right) = \frac{x/2}{R}$$

$$\implies x = 2R\sin\left(\frac{\theta}{2}\right)$$

Diff w.r.t t

$$\frac{dx}{dt} = R\cos\left(\frac{\theta}{2}\right)\frac{d\theta}{dt}$$

Here $R = 20cm, \quad \theta = \left(\frac{2\pi}{12}\right)(4), \quad \frac{d\theta}{dt} = \frac{2\pi}{60}$

$$\frac{dx}{dt} = \frac{\pi}{3}cm/sec$$

Problem 6

In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP(t)}{dt} = r_0 \left(1 - \frac{P(t)}{P_c}\right) P(t) - \beta P(t)$$

where r_0 is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the carrying capacity), and β is the percentage of the population that is harvested.

(a) What $\frac{dP}{dt}$ value of corresponds to a stable population?

Solution:When the rate of change of fish population is zero then the population of fish is become stable.i.e. if $\dfrac{dP}{dt} = 0$, then population is stable.

(b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.

Solution:

$$\frac{dP}{dt} = 0 \implies r_0 \left(1 - \frac{P(t)}{P_c}\right) P(t) - \beta P(t) = 0$$

$$\left(r_0 \left(1 - \frac{P(t)}{P_c}\right) - \beta\right) P(t) = 0$$

Since $P(t) \neq 0$ then

$$\left(r_0 \left(1 - \frac{P(t)}{P_c}\right) - \beta\right) = 0 \implies P(t) = P_c \left(1 - \frac{\beta}{r_0}\right)$$

Here $P_c = 10,000, r_0 = 0.05(5\%), \beta = 0.04(4\%)$ then $P(t) = 2000$

(c) What happens if β is raised to 5%?

Solution:

Since β is increased to 5% then only $P(t) = 0$ will be the stable population in the long run.

Problem 7

Find the absolute extrema of the following functions. State the possible critical points in the given interval.

[Note: You don't need to look anywhere online about how to differentiate the absolute value functions and apply all those strange methods. It cannot be emphasized enough that drawing a graph is always helpful. So to find their critical points, just draw their graphs.]

(a) $f(x) = \frac{2}{3}x^2 - 5 \quad -2 \leq x \leq 3$

Solution:
$f'(x) = \frac{4}{3}x = 0, \implies \quad x = 0$
Critical points: $x = 0$ At critical points:

$f(0) = -5$

At endpoints:

$f(-2) = -\frac{7}{3}, \quad f(3) = 1$

Abs Max: $f(3) = 1$ Abs Min: $f(0) = -5$

(b) $f(x) = \sqrt[3]{x} \quad -1 \leq x \leq 8$

Solution:
$f'(x) = \frac{1}{3}x^{2/3}$
Critical points: $x = 0$ when $f'(x)$ is undefined

At critical points:

$f(0) = 0$

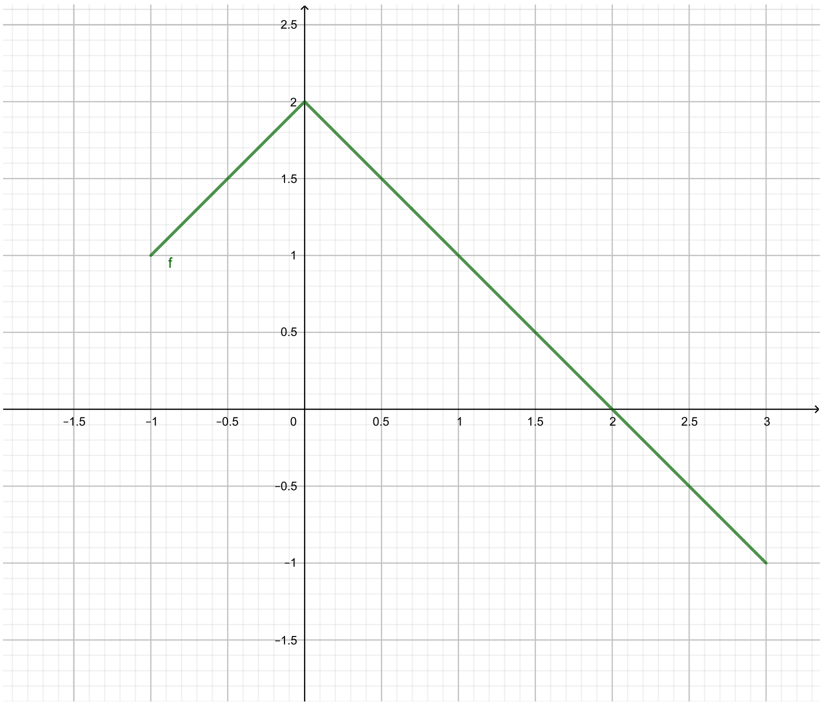
At endpoints:

$f(-1) = -1, \quad f(8) = 2$

Abs Max: $f(-1) = -1$ Abs Min: $f(8) = 2$

(c) $f(x) = 2 - |x| \qquad -1 \leq x \leq 3$

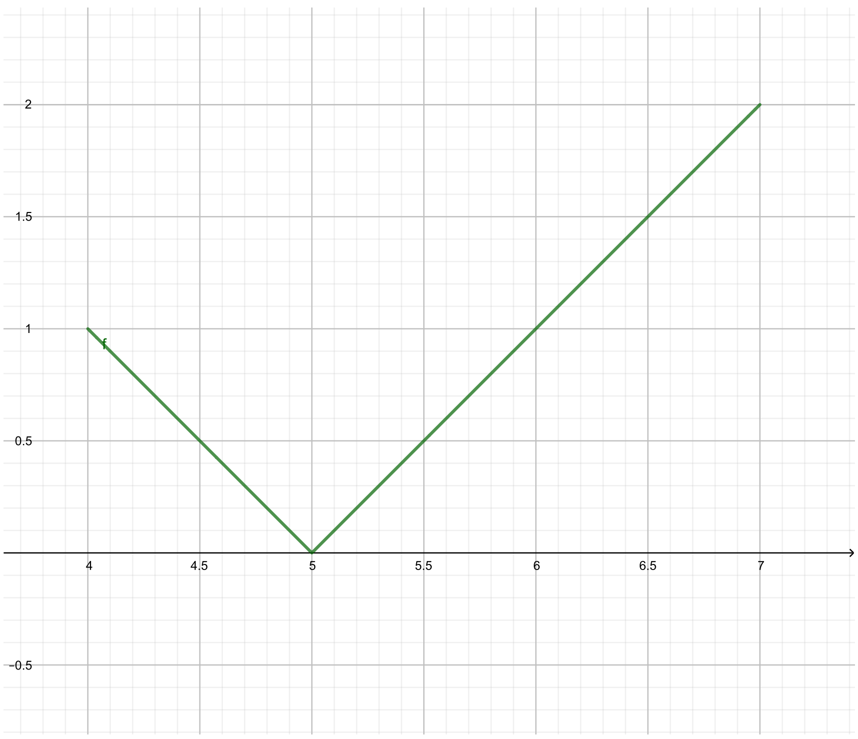
Solution:



Using the graph, Abs Max: $f(0) = 2$ Abs Min: $f(3) = -1$

(d) $f(x) = |x - 5| \qquad 4 \leq x \leq 7$

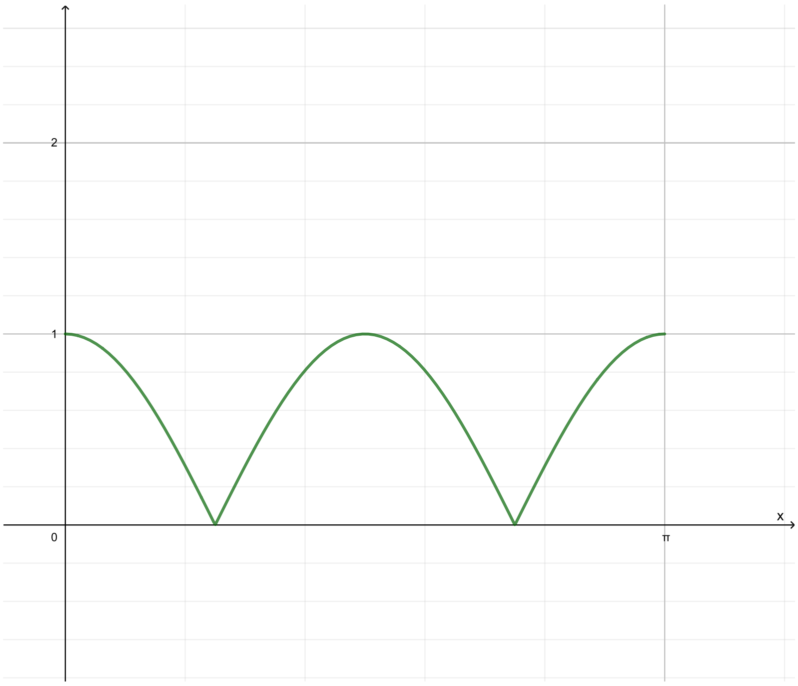
Solution:



Using the graph, Abs Max: $f(7) = 2$ Abs Min: $f(5) = 0$

(e) $f(x) = |\cos(2x)| \quad 0 \leq x \leq \pi$

Solution:



Using the graph, Abs Max: $f(0) = f(\frac{\pi}{2}) = f(\pi) = 1$ Abs Min: $f(\frac{\pi}{4}) = f(\frac{3\pi}{4}) = 0$

Problem 8

For the following functions

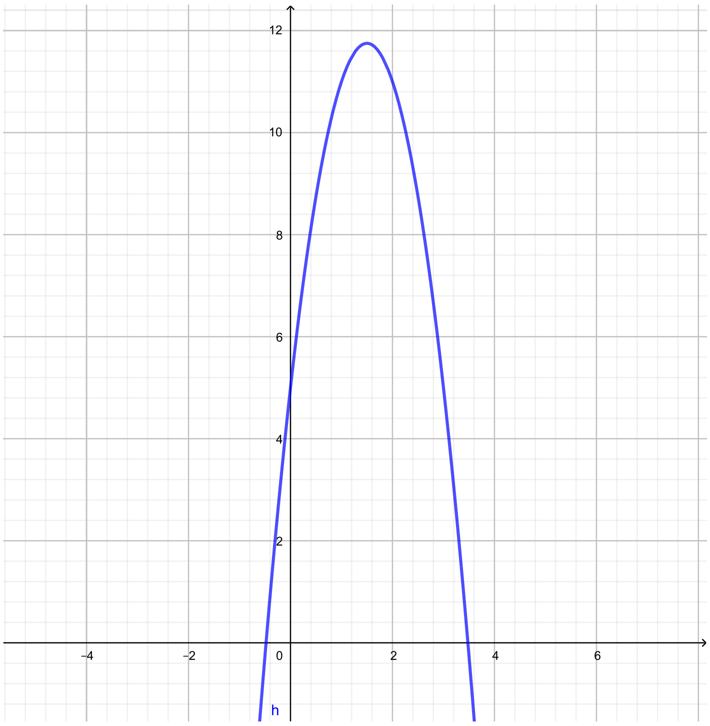
(i) Using the first derivative function test for extremum, find the interval where the derivative is increasing or decreasing then decide about the local extremum of the functions and classify them.

(ii) Sketch the graph of the functions using above information.

(a) $g(t) = -3t^2 + 9t + 5$
Solution:
$g'(t) = -6t + 9 = 0 \implies t = \frac{3}{2}$

	$(-\infty, \frac{3}{2})$	$(\frac{3}{2}, \infty)$
$f'(x)$	$+$	$-$
$f(x)$	Increasing	Decreasing

$f'(x)$ changes sign from $+ve$ to $-ve$ at $x = \frac{3}{2}$ so it is local maximum.



(b) $f(x) = \frac{x^4}{4} - x^3 - 3x^2 + 8x$

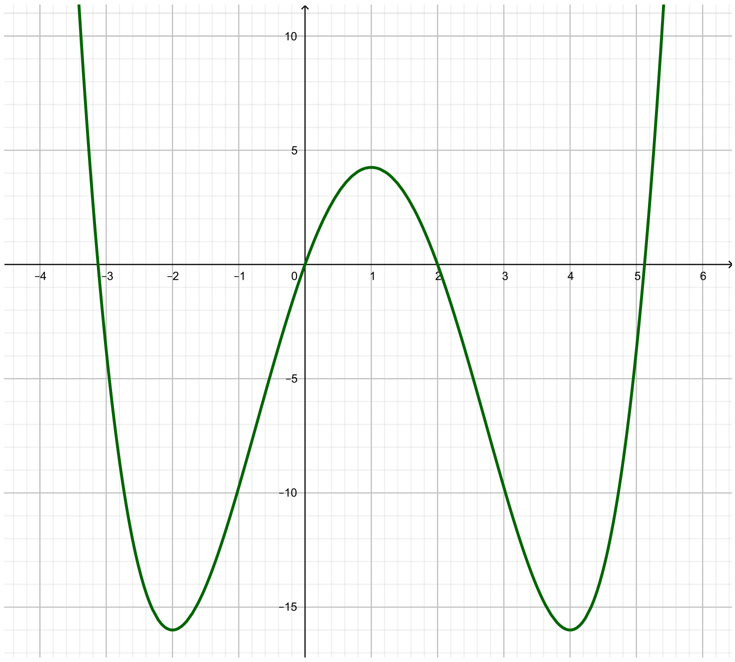
Solution:

$$f'(x) = x^3 - 3x^2 - 6x + 8 = (x-4)(x-1)(x+2) = 0 \implies x = -2,\ 1,\ 4$$

	$(-\infty, -2)$	$(-2, 1)$	$(1, 4)$	$(4, \infty)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	Decreasing	Increasing	Decreasing	Decreasing

$f'(x)$ changes sign from $+ve$ to $-ve$ at $x = 1$, so it is local maximum.

$f'(x)$ changes sign from $-ve$ to $+ve$ at $x = -2,\ 4$ so these are local minimum.



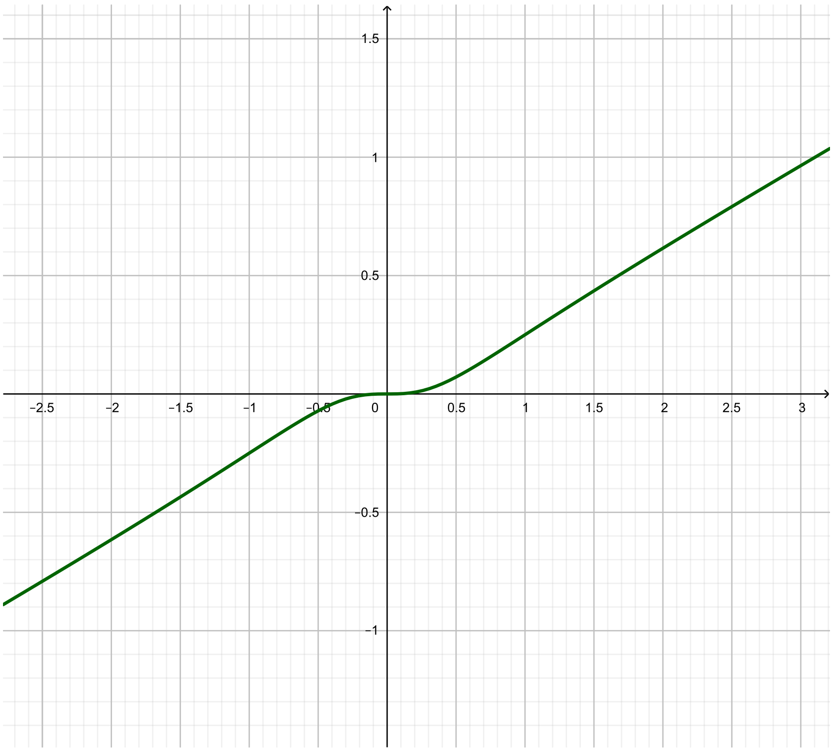
(c) $f(x) = \dfrac{x^3}{3x^2+1}$

Solution:

$f'(x) = \frac{3(x^4+x^2)}{(3x^2+1)^2} = 0 \implies x = 0$

	$(-\infty, 0)$	$(0, \infty)$
$f'(x)$	$+$	$+$
$f(x)$	Increasing	Increasing

$f'(x)$ never changes the sign at $x = 0$, so it is neither maximum nor minimum.



(d) $f(x) = x^{2/3}(x^2 - 4)$

Solution:

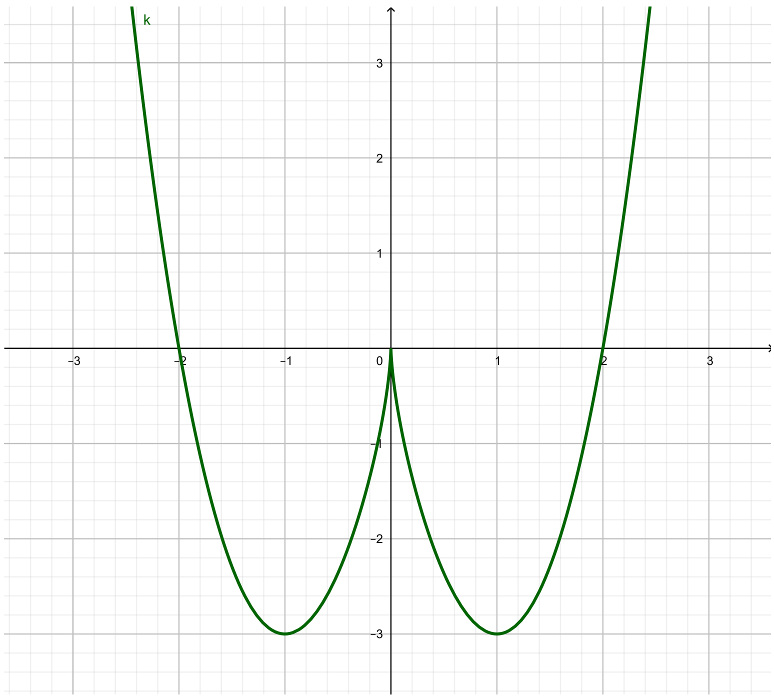
$f'(x) = \left\{ \frac{8(x^2-1)}{3\sqrt[3]{x}} \right\} = 0 \implies x = -1,\ 1$

Also $f'(x)$ is undefined at $x = 0$ so it will also be considered as critical point.

	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	Decreasing	Increasing	Decreasing	Increasing

$f'(x)$ changes sign from $+ve$ to $-ve$ at $x = 0$, so it is local maximum.

$f'(x)$ changes sign from $-ve$ to $+ve$ at $x = -1,\ 1$ so these are local minimum.



Problem 9

Show that the equation $x = \cos(x)$ has exactly one solution.

Solution: Define $f(x) = x - \cos(x)$ then $f(0) = 0 - \cos(0) = 0 - 1 < 0$ and $f(\frac{\pi}{2}) = \frac{\pi}{2} - 0 > 0$. Also $f(x)$ is continuous then using intermediate value theorem that there exist at least one $c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$.

Roll's theorem: Let $f(x)$ be continuous on closed interval $[a, b]$ and differentiable on open interval (a, b). If $f(a) = f(b)$ then there is a number c such that $f'(c) = 0$.

Now, Assume that there exists two roots a and $b \in (0, \frac{\pi}{2})$ such that $f(a) = 0$ and $f(b) = 0$. $f(x)$ is differentiable in the interval then by Roll's theorem guarantee that there exist a value c_1 such that

$$f'(c_1) = 0$$

$$1 + \sin(c_1) = 0, \quad c_1 \notin (0, \frac{\pi}{2})$$

its been found that $c_1 = \frac{3\pi}{2}$ which is not in the desired interval.So our assumption is wrong that there are two roots $\implies a = b$ Hence, $f(x)$ has only one root.

Problem 10

The height above ground of object moving vertically is given as

$$s = -16t^2 + 96t + 112$$

with s in feet and t is in seconds. Find

(a) the object's velocity when $t = 0$
Solution:
$v(t) = s'(t) = -32t + 96 \implies v(0) = 96$

(b) its maximum height and when it occurs

Solution:
Finding the critical points $s'(t) = -32t + 96 = 0 \implies t = 3$

	$(0, 3)$	$(3, \infty)$
$s'(t)$	$+$	$-$
$s(t)$	Increasing	Decreasing

so maximum height will at $t = 3$ sec as $s'(t)$ changes sign from +ve to -ve. The maximum height will be $s(3) = 256$ $feet$

(c) its velocity when $s = 0$

Solution:
When $s = 0$ the object will hit the ground.

$$s = -16t^2 + 96t + 112 = 0$$
$$-16(t - 7)(t + 1) = 0$$
$$t = 7, \quad t = -1$$

So object will hit at $t = 7$ sec on ground. The velocity of the object will be

$$v = -32(7) + 96 = -128$$

The $-ve$ sign shows that it is moving vertically downward.