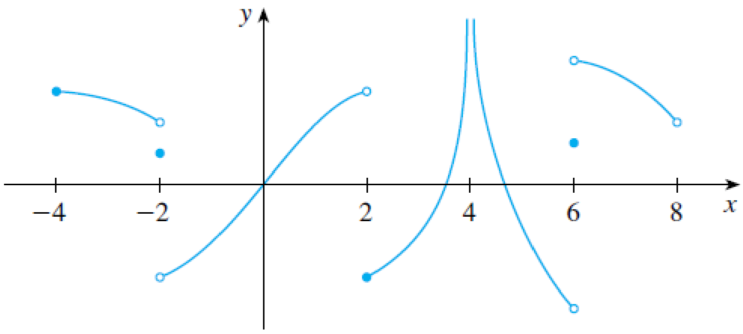
MT110: Calculus and Analytic Geometry

Homework 3

Due: Fri, Oct 26, 2:00 pm *Fall 2018*

Problem 1

For each of the relevant intervals in the graph of $g(x)$ shown below, state whether the function is continuous, continuous from the left only, continuous from the right only or neither.
Divide the graph into intervals.



(a) Continous on [-4,2) i.e. continous from right at -4.

(b) Continous on (-2,2) i.e. continous from neither side

(c) Continous on [2,4) i.e. continous from right at 2

(d) Continous on (4,6), i.e. continous from neither side

(e) Continous on (6,8) i.e. continous from neither side

Problem 2

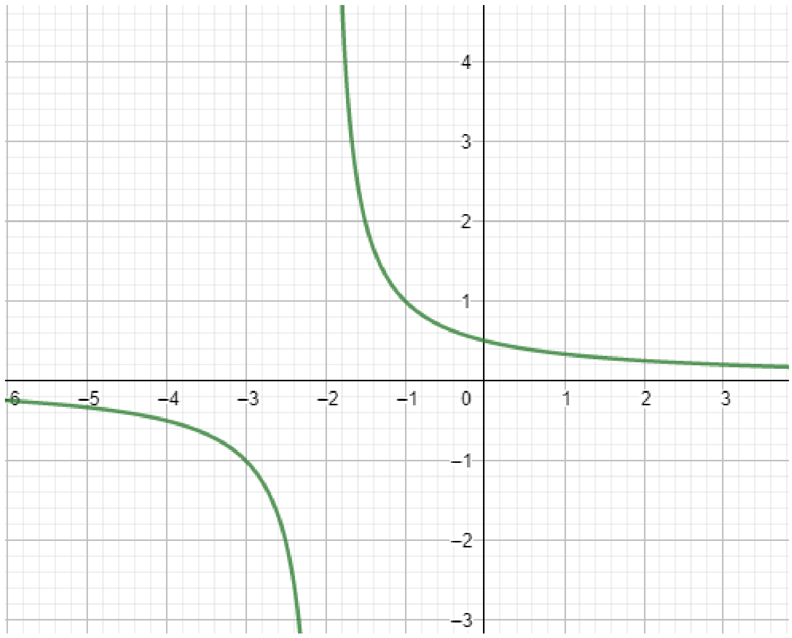
For each of the following functions

(i) Sketch the graph of the function.

(ii) Explain why the function is discontinuous at the given value $x = x_0$. State the type of discontinuity (removable, infinite or jump).

(iii) At which of these values f is continuous from the right, from the left, or neither?

(a) $f(x) = \dfrac{1}{x+2}$, $x_0 = -2$

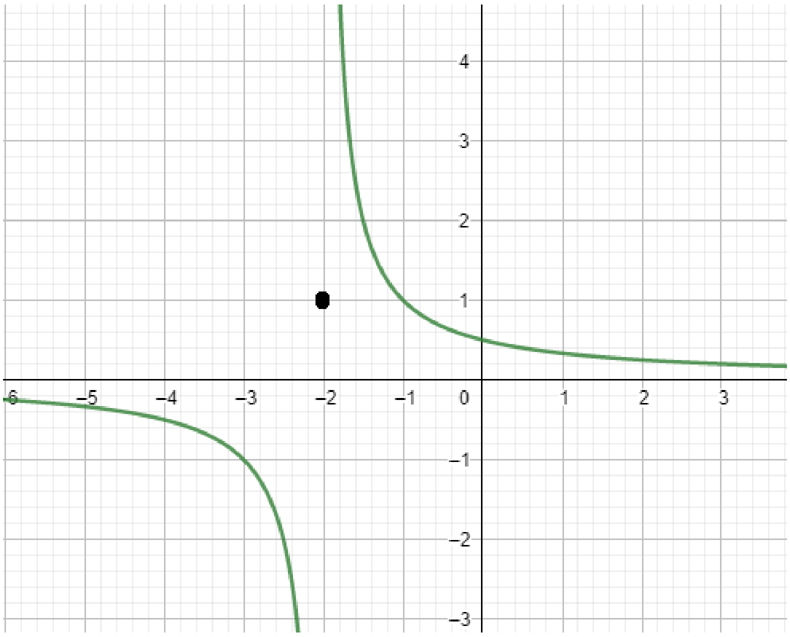


(i)

(ii) Infinite Discontinuity

(iii) Neither continuous from right nor left.

(b) $f(x) = \begin{cases} \dfrac{1}{x+2}, & \text{if } x \neq -2 \\ 1, & \text{if } x = -2 \end{cases}$, $x_0 = -2$

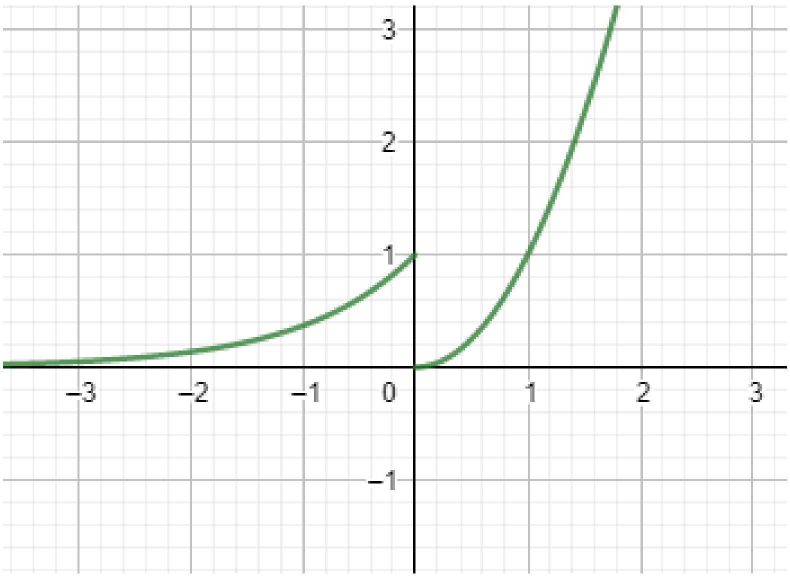


(i)

(ii) Infinite Discontinuity

(iii) Neither continuous from right nor left.

(c) $f(x) = \begin{cases} e^x, & \text{if } x < 0 \\ x^2, & \text{if } x \geq 0 \end{cases}$, $x_0 = 0$

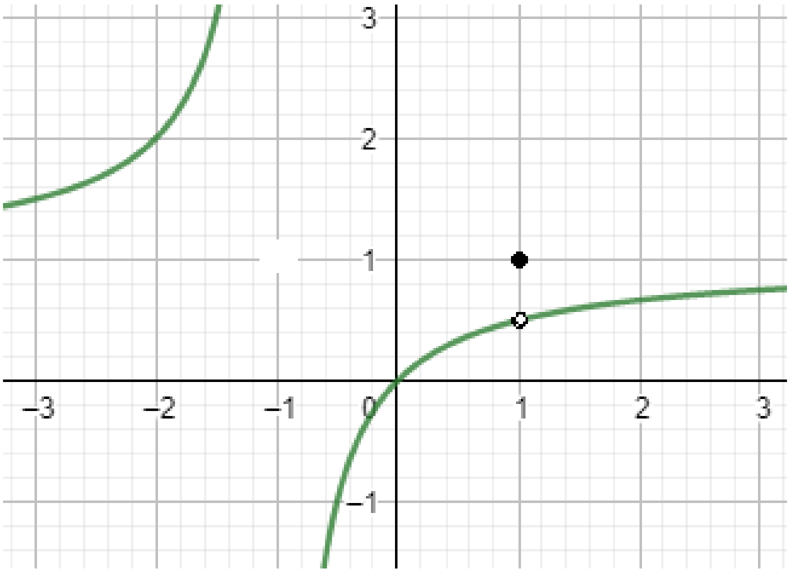


(i)

(ii) Jump Discontinuity

(iii) Continous from the right

(d) $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1}, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$, $x_0 = 1$

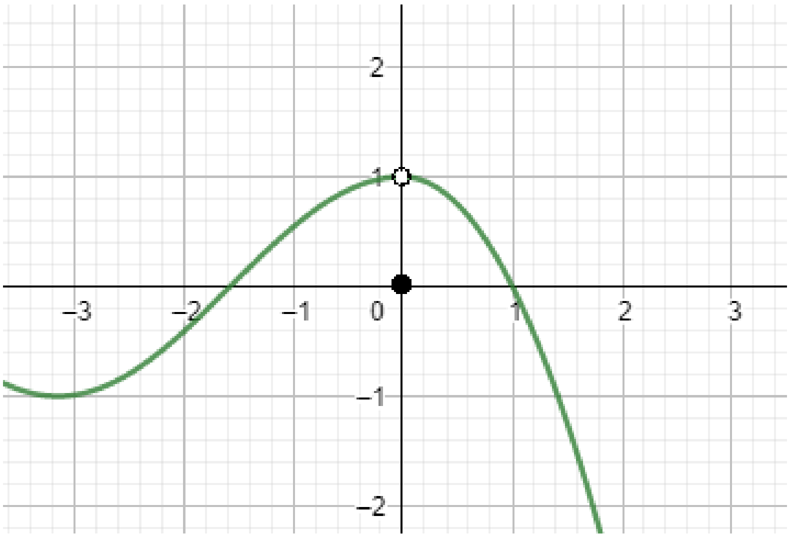


(i)

(ii) Infinite Discontinuity

(iii) Neither continuous from right nor left.

(e) $f(x) = \begin{cases} \cos x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1 - x^2, & \text{if } x > 0 \end{cases}$, $x_0 = 0$



(i)

(ii) Removable Discontinuity

(iii) Neither continuous from right nor left.

Problem 3

Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x\to 2} (3f(x) + f(x)g(x)) = 36$. Find $f(2)$.

$$3f(2) + f(2)g(2) = 36$$
$$3f(2) + 6f(2) = 36$$
$$9f(2) = 36$$
$$f(2) = 4$$

Problem 4

The function $f(x) = \dfrac{x^4 - 1}{x - 1}$ has a removable discontinuity at $x = a$.

(a) What is the value of a?

(b) Find a function g that equals f for $x \neq a$ but is continuous at $x = a$.

(a)

$$a = 1$$

(b) To find the continuous function at $x = 1$, we will compute its limit at $x = 1$.

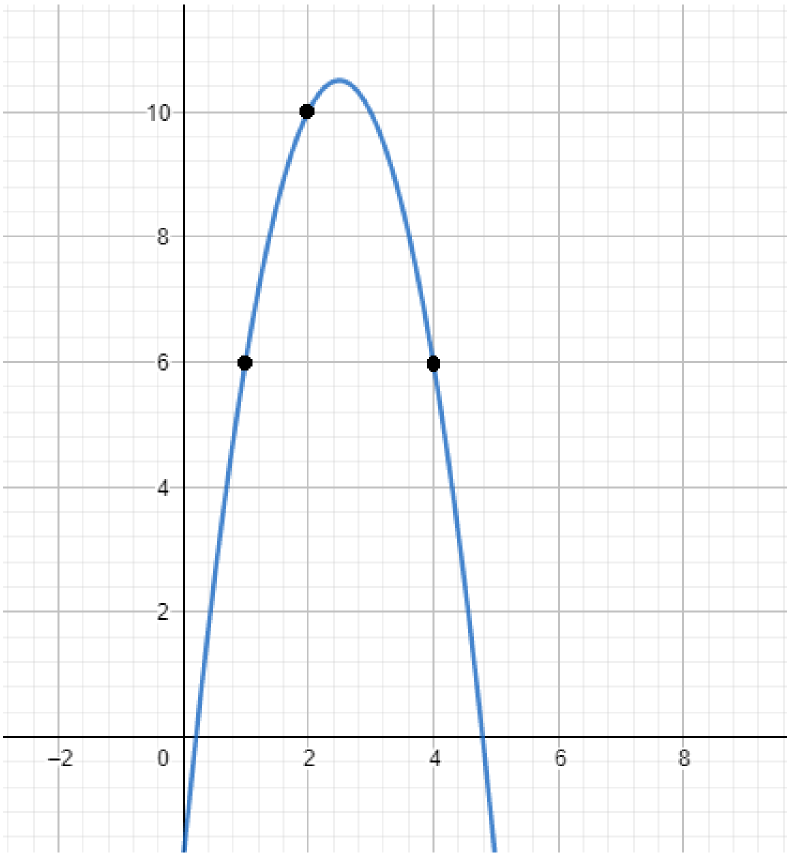
$$\lim_{x\to 1} \frac{x^4 - 1}{x - 1}$$
$$= \lim_{x\to 1} \frac{(x^2 - 1)(x^2 + 1)}{x - 1}$$
$$= \lim_{x\to 1} \frac{(x + 1)(x - 1)(x^2 + 1)}{x - 1}$$
$$= \lim_{x\to 1} (x + 1)(x^2 + 1) = 4$$
$$g(x) = \begin{cases} \dfrac{x^4 - 1}{x - 1}, & \text{if } x \neq 1 \\ 4, & \text{if } x = 1 \end{cases}$$

Problem 5

(a) Suppose f is continuous on [0 5] and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 10$, explain why $f(3) > 6$, using a graph and referring to an appropriate theorem.

(b) Find all the intervals on which $g(x) = \sqrt{\dfrac{2x + 3}{x - 2}}$ is continuous.

(a) By intermediate value theorem, f(x) is continuous on interval $[1, 4]$ where $f(1) = 6$, $f(4) = 6$n and f(2)=10. So, f(3) must be greater then 6.



(b) Square-root of $\frac{2x+3}{x-2}$ to exist there are two cases for which $\frac{2x+3}{x-2} > 0$

Case 1:

$$2x + 3 > 0$$
$$x > -\frac{3}{2}$$
$$x - 2 > 0$$
$$x > 2$$

So, $x > 2$

Case 2:

$$2x + 3 < 0$$
$$x < -\frac{3}{2}$$
$$x - 2 < 0$$
$$x < 2$$

So, $x < -\frac{3}{2}$ By case 1 and case 2, f(x) is continuous on the interval $(-\infty, -\frac{3}{2}) \cup (2, \infty)$

Problem 6

Evaluate the following limits and justify each step by indicating the appropriate properties of limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8}$

$$= \lim_{x\to\infty} \frac{3 - \frac{1}{x} + \frac{4}{x^2}}{2 + \frac{5}{x} - \frac{8}{x^2}}$$

$$= \frac{3 - \lim\limits_{x\to\infty} \frac{1}{x} + \lim\limits_{x\to\infty} \frac{4}{x^2}}{2 + \lim\limits_{x\to\infty} \frac{5}{x} - \lim\limits_{x\to\infty} \frac{8}{x^2}}$$

$$= \frac{3 - 0 + 0}{2 + 0 - 0}$$

$$= \frac{3}{2}$$

(b) $\displaystyle\lim_{x\to-\infty} \frac{x^3 - x + 1}{x^2 + x - 2}$

$$= \lim_{x\to-\infty} \frac{x - \frac{1}{x} + \frac{1}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}}$$

$$= \frac{\lim\limits_{x\to-\infty} x - \lim\limits_{x\to-\infty} \frac{1}{x} + \lim\limits_{x\to-\infty} \frac{1}{x^2}}{1 + \lim\limits_{x\to-\infty} \frac{1}{x} - \lim\limits_{x\to-\infty} \frac{2}{x^2}}$$

$$= \frac{-\infty - 0 + 0}{1 + 0 - 0}$$

$$= -\infty$$

(c) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{2 - x^3 + 16x^6}}{1 + 4x^2 + 2x^3}$

$$= \lim_{x\to-\infty} \frac{-x^3\sqrt{\frac{2}{x^6} - \frac{1}{x^3} + 16}}{x^3\left(\frac{1}{x^3} + \frac{4}{x} + 2\right)}$$

$$= \lim_{x\to-\infty} -\frac{\sqrt{\lim\limits_{x\to-\infty} \frac{2}{x^6} - \lim\limits_{x\to-\infty} \frac{1}{x^3} + 16}}{\lim\limits_{x\to-\infty} \frac{1}{x^3} + \lim\limits_{x\to-\infty} \frac{4}{x} + 2}$$

$$= -\frac{\sqrt{0 - 0 + 16}}{0 + 0 + 2} = -2$$

(d) (d) $\displaystyle\lim_{x\to\infty} \sqrt{\frac{2 - x^2 + 12x^3}{1 + 4x^2 + 3x^4}}$

$$= \lim_{x\to\infty} \sqrt{\frac{\frac{2}{x^4} - \frac{1}{x^3} + \frac{12}{x^2}}{\frac{1}{x^4} + \frac{4}{x^2} + 3}}$$

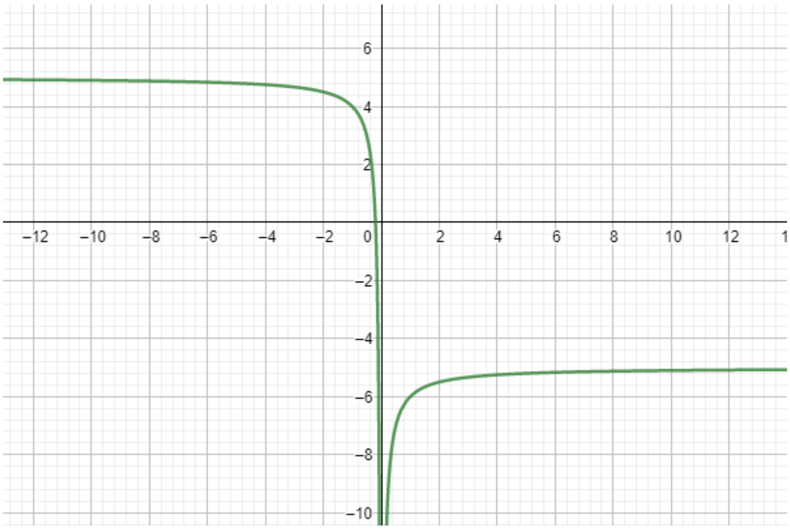
$$= \sqrt{\frac{\lim\limits_{x\to\infty} \frac{2}{x^4} - \lim\limits_{x\to\infty} \frac{1}{x^3} + \lim\limits_{x\to\infty} \frac{12}{x^2}}{\lim\limits_{x\to\infty} \frac{1}{x^4} + \lim\limits_{x\to\infty} \frac{4}{x^2} + 3}}$$

$$= \frac{0 + 0 + 0}{0 + 0 + 3} = 0$$

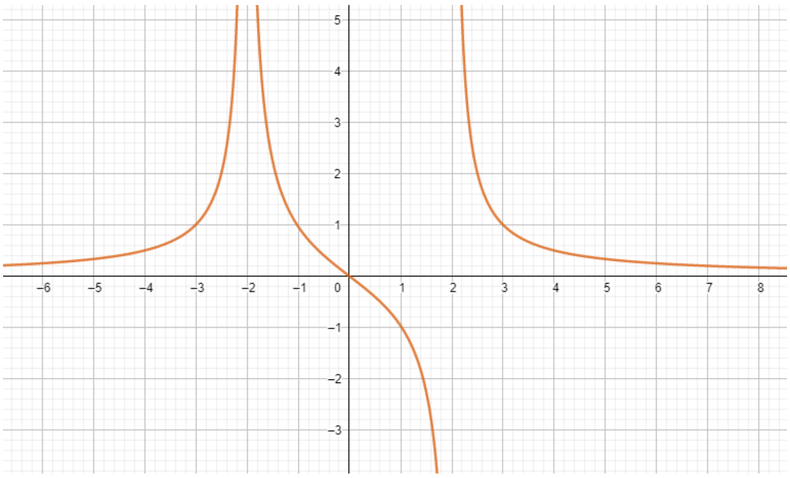
Problem 7

Sketch the graph of an example of a function f that satisfies all of the given conditions.

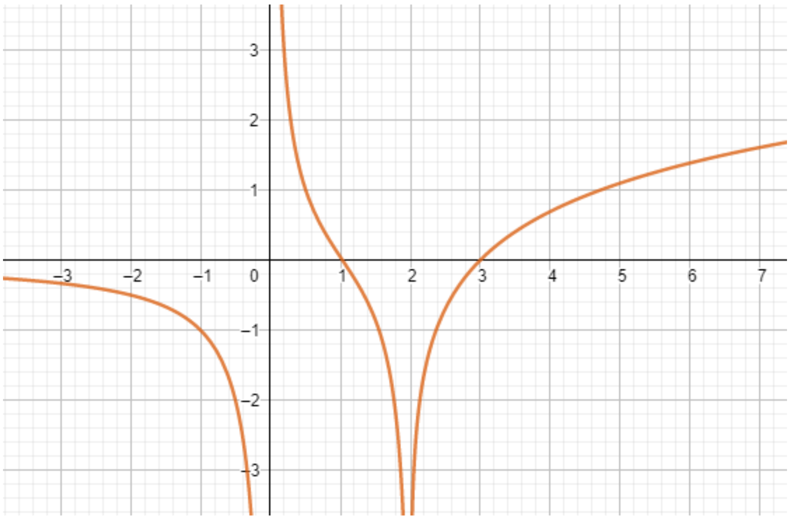
(a) $\lim_{x \to 0} f(x) = -\infty, \quad \lim_{x \to -\infty} f(x) = 5, \quad \lim_{x \to \infty} f(x) = -5$



(b) $\lim_{x \to -2} f(x) = \infty, \quad \lim_{x \to 2^+} f(x) = \infty, \quad \lim_{x \to 2^-} f(x) = -\infty,$
$\lim_{x \to -\infty} f(x) = 0, \quad \lim_{x \to \infty} f(x) = 0, \quad f(0) = 0$



(c) $\lim_{x \to 2} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = 0$,
$\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = -\infty$



Problem 8

Make a rough sketch of the curve $y = x^n$ (for $n \in \mathbb{Z}$) for the following five cases:

(a) $n = 0$ (b) $n < 0$, n is odd (c) $n > 0$, n is even

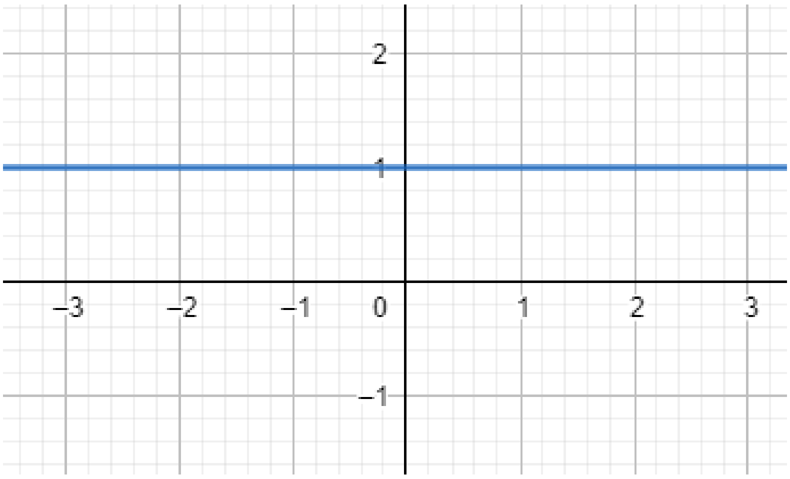
Then use these sketches to find the following limits for each case (a) to (c).

(i) $\lim_{x \to 0^+} x^n$

(ii) $\lim_{x \to 0^-} x^n$

(iii) $\lim_{x \to \infty} x^n$

(iv) $\lim_{x \to -\infty} x^n$

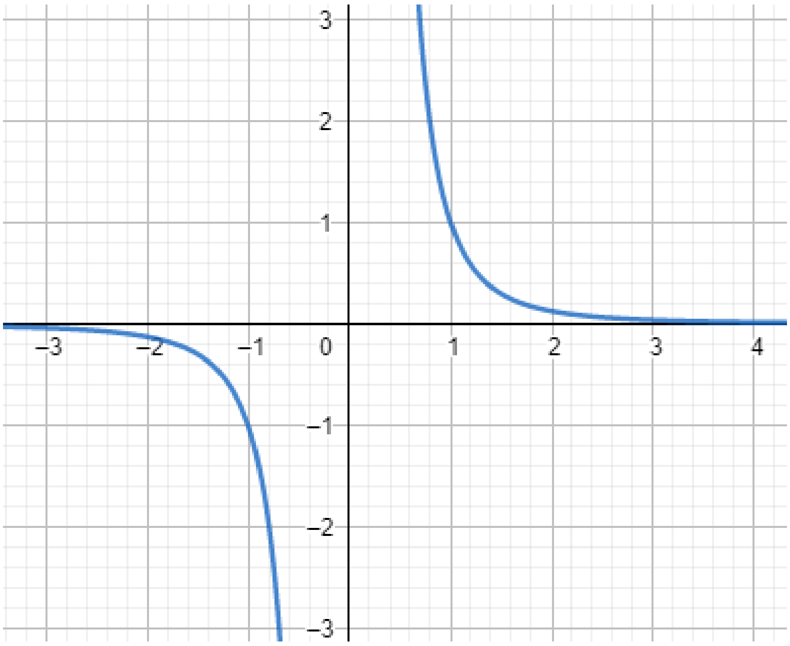


(a) (i) $\lim_{x \to 0^+} x^n = 1$

(ii) $\lim_{x \to 0^-} x^n = 1$

(iii) $\lim_{x \to \infty} x^n = 1$

(iv) $\lim_{x \to -\infty} x^n = 1$

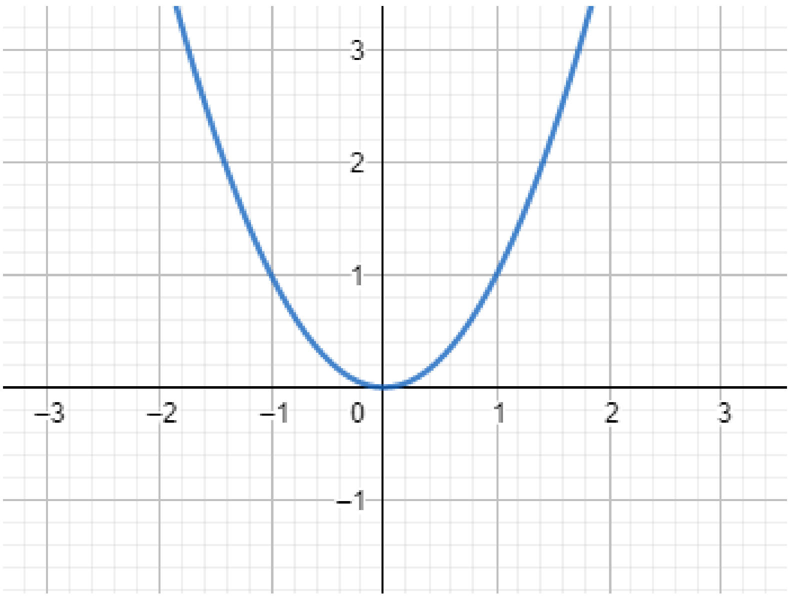


(b) (i) $\lim_{x\to 0^+} x^n = \infty$

(ii) $\lim_{x\to 0^-} x^n = -\infty$

(iii) $\lim_{x\to \infty} x^n = 0$

(iv) $\lim_{x\to -\infty} x^n = 0$



(c) (i) $\lim_{x\to 0^+} x^n = 0$

(ii) $\lim_{x\to 0^-} x^n = 0$

(iii) $\lim_{x\to \infty} x^n = \infty$

(iv) $\lim_{x\to -\infty} x^n = \infty$

Problem 9

A tank contains 15000 L of pure water. Brine that contains 20 g of salt per liter of water is pumped into the tank at a rate of 50 L per minute.

(a) Show that the concentration (in grams per liter) of salt after t minutes is

$$C(t) = \frac{20t}{300+t}$$

(b) Using the result in (a), predict what would happen to the concentration after a very long time?

(a) $C(t)$ =gs of salt in the tank at time t minutes
$C'(t)$ =(rate of salt going in)-(rate of salt going out)

$$rate\ in = \frac{20\ g\ of\ salt\ per}{liter} \times \frac{50\ liters}{min} = \frac{1000gs\ of\ salt}{min}$$

rate in at time t minutes $= 1000t$
How much solution is in the tank? 15000 liters.
How much solution is in the tank after 1 minute?

$$= 15000 + 50\ (liters\ flowing\ in\ per\ minute)\ = 15000 + 50 = 15050\ liters.$$

how much solution is in the tank after 2 minute?

$$= 15000 + 50\ (liters\ flowing\ in\ per\ minute)\ \times 2 = 15000 + 100 = 15100\ liters.$$

how much solution is in the tank after t minute?

$$= 15000\ \ + 50t$$

$$C(t) = \frac{1000t}{15000+50t}$$

$$C(t) = \frac{20t}{300+t}$$

(b) After a very long time let t approach to infinity.

$$\lim_{t\to\infty} C(t)$$

$$= \lim_{t\to\infty} \frac{20t}{300+t}$$

$$= \lim_{t\to\infty} \frac{20}{\frac{300}{t}+1}$$

$$= \frac{20}{\frac{300}{\infty}+1}$$

$$= 20$$

Problem 10

Use the precise definition of a limit to prove that

(a) $\lim_{x \to -\infty} \left(\frac{1}{x} + 2 \right) = 2$

$$x < N$$

Let $N = \frac{-1}{\epsilon}$

$$x < \frac{-1}{\epsilon}$$
$$\implies \frac{1}{x} > -\epsilon$$
$$\implies \frac{1}{x} + 2 - 2 > -\epsilon$$
$$\implies \left| \frac{1}{x} + 2 - 2 \right| < \epsilon$$
$$|f(x) - L| < \epsilon$$

(b) $\lim_{x \to \infty} \ln x = \infty$

Let $N = e^M$

$$x > e^M$$
$$\implies \ln x > \ln e^M$$
$$\implies y > M$$

Problem 11

For each of the following curves,

(a) $y = 4x - 3x^2, \quad (2, -4)$ (b) $y = \sqrt{x}, \quad (1, 1)$ (c) $y = \frac{2x+1}{x+2}, \quad (1, 1)$

(i) Using the limit definition of the slope of a tangent line, find the slope of the tangent to the curve at the given point.

(ii) Find an equation of the tangent line to the curve at the given point.

(iii) Sketch the curve and tangent on the same graph.

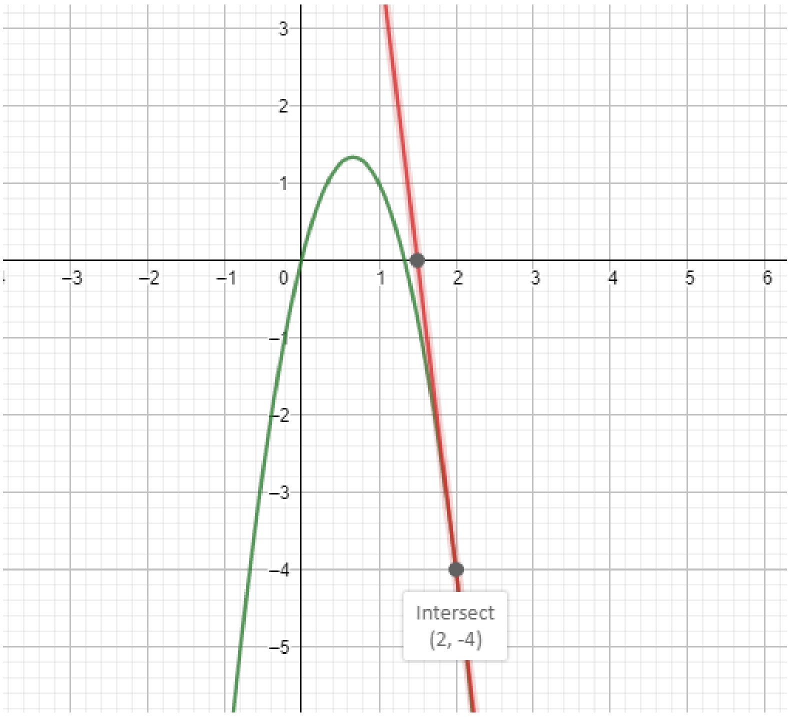
(a) $y = 4x - 3x^2, \quad (2, -4)$

(i) Slope of tangent at point $P(a, f(a))$:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$m = \lim_{x \to 2} \frac{f(x) - (-4)}{x - 2}$$
$$m = \lim_{x \to 2} \frac{-3x^2 + 4x - (-4)}{x - 2}$$

$$m = \lim_{x \to 2} \frac{-3x^2 + 6x - 2x + 4)}{x - 2}$$
$$m = \lim_{x \to 2} \frac{-3x(x-2) - 2(x-2))}{x - 2}$$
$$m = \lim_{x \to 2} \frac{(-3x-2)(x-2))}{x - 2}$$
$$m = (-3(2) - 2) = -8$$

(ii) Equation of the tangent line:
$y - (-4) = -8(x - 2)$ $y = -8x + 12$



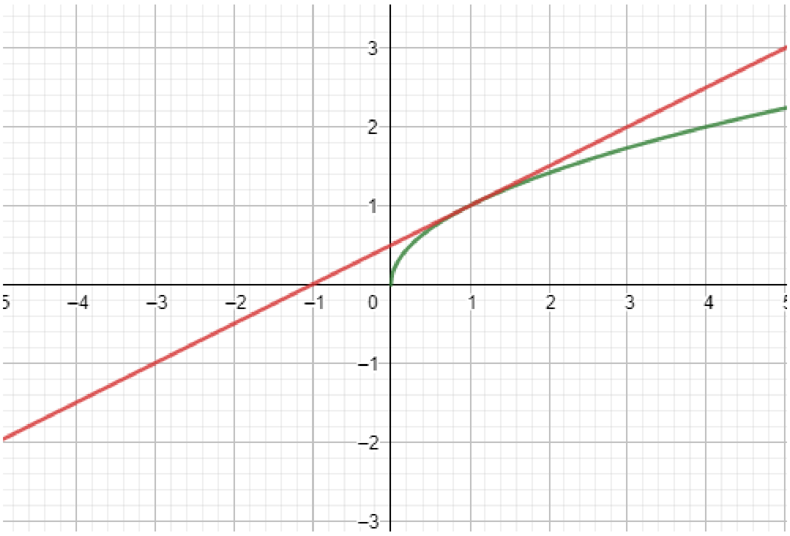
(iii)

(b) $y = \sqrt{x}, \quad (1, 1)$

(i) Slope of tangent at point $P(a, f(a))$:
$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$m = \lim_{x \to 1} \frac{\sqrt{x} - (1)}{x - 1}$$
$$m = \lim_{x \to 1} \frac{\sqrt{x} - (1)}{(\sqrt{x} - 1)(\sqrt{x} + 1)}$$
$$m = \lim_{x \to 1} \frac{1}{\sqrt{x} + (1)}$$
$$m = \frac{1}{2}$$

(ii) Equation of the tangent line:
$y - 1 = \frac{1}{2}(x - 1)$
$y = \frac{1}{2}x + \frac{1}{2}$

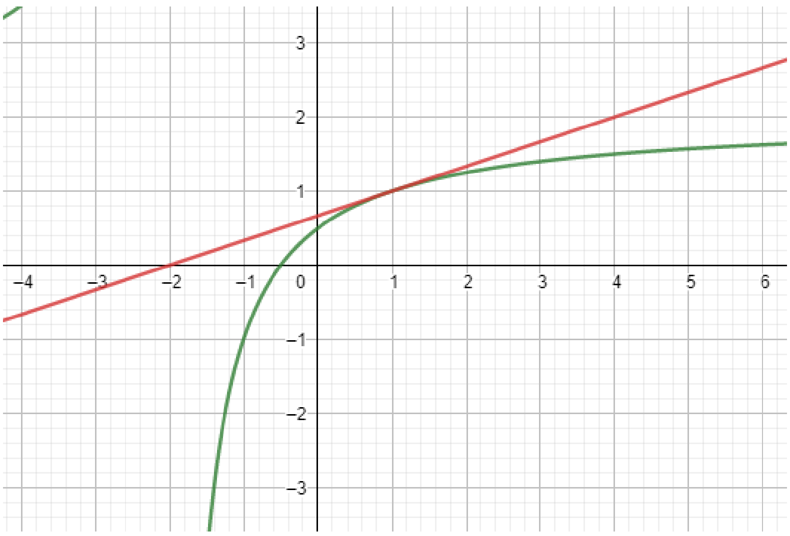
(iii)



(c) $y = \dfrac{2x+1}{x+2}$, $(1, 1)$

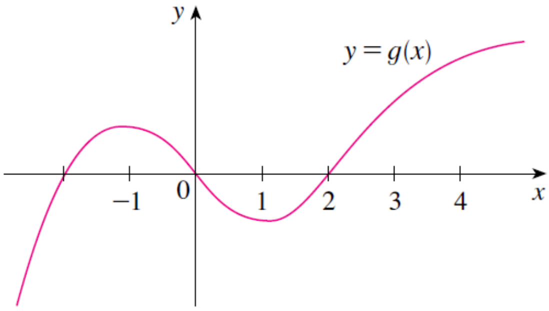
(i) Slope of tangent at point $P(a, f(a))$:
$m = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$
$m = \lim_{x \to 1} \dfrac{\frac{2x+1}{x+2} - (1)}{x - 1}$
$m = \lim_{x \to 1} \dfrac{x-1}{x+2} \times \dfrac{1}{x-1}$
$m = \frac{1}{3}$

(ii) Equation of the tangent line:
$y - 1 = \frac{1}{3}(x - 1)$
$y = \frac{1}{3}x + \frac{2}{3}$



(iii)

Problem 12



For the function $g(x)$ whose graph is given, arrange the following values in increasing order and explain your reasoning: $g'(-2)$, $g'(-1)$, $g'(0)$, $g'(2)$, $g'(4)$

$g'(0)$, $g'(-1)$, $g'(4)$, $g'(2)$, $g'(-2)$

The tanget at 0 will have maximum negative slope.

At '-1' the tangent will be horizontal having a slope of zero.

Now tangents at g(4), g(-2) and g(2) wil all have positive slopes with g(-2) being the maximum, and g(4) being the minimum.

Problem 13

Find the derivative of the function using the limit definition of derivative (a.k.a. derivative from first principles). State the domain of the function and the domain of its derivative.

(a) $f(x) = \frac{1}{2}x - \frac{1}{3}$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}(x+h) - \frac{1}{3} - \frac{1}{2}(x) + \frac{1}{3}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}h}{h}$$

$$= \frac{1}{2}$$

(b) $f(t) = 5t - 9t^2$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{5(t+h) - 9(t+h)^2 - 5t + 9t^2}{h}$$

$$= \lim_{h\to 0} \frac{5t + 5h - 9t^2 - 9h^2 - 18th - 5t + 9t^2}{h}$$

$$= \lim_{h\to 0} \frac{5h - 9h^2 - 18th}{h}$$

$$= \lim_{h\to 0} 5 - 9h - 18t$$

$$= 5 - 18t$$

(c) $f(t) = \dfrac{1}{\sqrt{t}}$

$$= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{t+h}} - \dfrac{1}{\sqrt{t}}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{t+h}} \times \dfrac{\sqrt{t}}{\sqrt{t}} - \dfrac{1}{\sqrt{t}}\dfrac{\sqrt{t+h}}{\sqrt{t+h}}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{\sqrt{t} - \sqrt{t+h}}{\sqrt{t}\sqrt{t+h}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{t} - \sqrt{t+h}}{h\sqrt{t}\sqrt{t+h}}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{t} - \sqrt{t+h}}{h\sqrt{t}\sqrt{t+h}}\right)\left(\frac{\sqrt{t} + \sqrt{t+h}}{\sqrt{t} + \sqrt{t+h}}\right)$$

$$= \lim_{h\to 0} \frac{t - t - h}{h\sqrt{t}\sqrt{t+h}(\sqrt{t} + \sqrt{t+h})}$$

$$= \frac{-1}{\sqrt{t}\sqrt{t}(\sqrt{t} + \sqrt{t})}$$

$$= \frac{-1}{2t^{\frac{3}{2}}}$$

(d) $g(x) = \sqrt{9-x}$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{9-x-h} - \sqrt{9-x}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{9-x-h} - \sqrt{9-x}}{h}\right)\left(\frac{\sqrt{9-x-h} + \sqrt{9-x}}{\sqrt{9-x-h} + \sqrt{9-x}}\right)$$

$$= \lim_{h\to 0} \frac{9-x-h-9+x}{h(\sqrt{9-x-h} + \sqrt{9-x})}$$

$$= \frac{-1}{(\sqrt{9-x} + \sqrt{9-x})}$$

$$= \frac{-1}{(2\sqrt{9-x})}$$

(e) $G(t) = \dfrac{1-2t}{3+t}$

$$= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1-2t-2h}{3+t+h} - \dfrac{1-2t}{3+t}}{h}$$

$$= \lim_{h\to 0} \frac{(3+t)(1-2t-2h) - (3+t+h)(1-2t)}{(3+t+h)(3+t)h}$$

$$= \lim_{h\to 0} \frac{3-6t-6h+t-2t^2-2th+3+6t+2t^2-t-h+2th}{(3+t+h)(3+t)h}$$

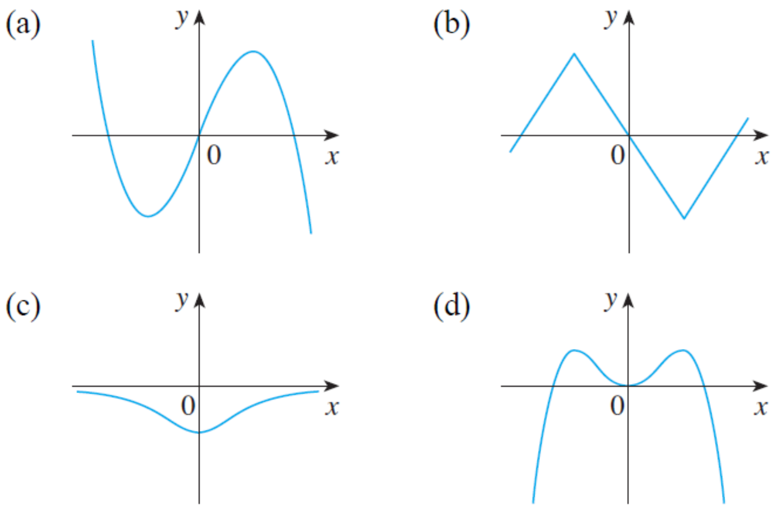
$$= \lim_{h\to 0} \frac{-7h}{(3+t+h)(3+t)h}$$
$$= \frac{-7}{(3+t)^2}$$

(f) $f(x) = x^{\frac{3}{2}}$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^{\frac{3}{2}} - x^{\frac{3}{2}}}{h}$$
$$= \lim_{h\to 0} \left(\frac{(x+h)^{\frac{3}{2}} - x^{\frac{3}{2}}}{h}\right)\left(\frac{(x+h)^{\frac{3}{2}} + x^{\frac{3}{2}}}{(x+h)^{\frac{3}{2}} + x^{\frac{3}{2}}}\right)$$
$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h((x+h)^{\frac{3}{2}} + x^{\frac{3}{2}})}$$
$$= \lim_{h\to 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h((x+h)^{\frac{3}{2}} + x^{\frac{3}{2}})}$$
$$= \lim_{h\to 0} \frac{h^2 + 3x^2 + 3xh}{((x+h)^{\frac{3}{2}} + x^{\frac{3}{2}})}$$
$$= \frac{3x^2}{((x)^{\frac{3}{2}} + x^{\frac{3}{2}})}$$
$$= \frac{3\sqrt{x}}{2}$$

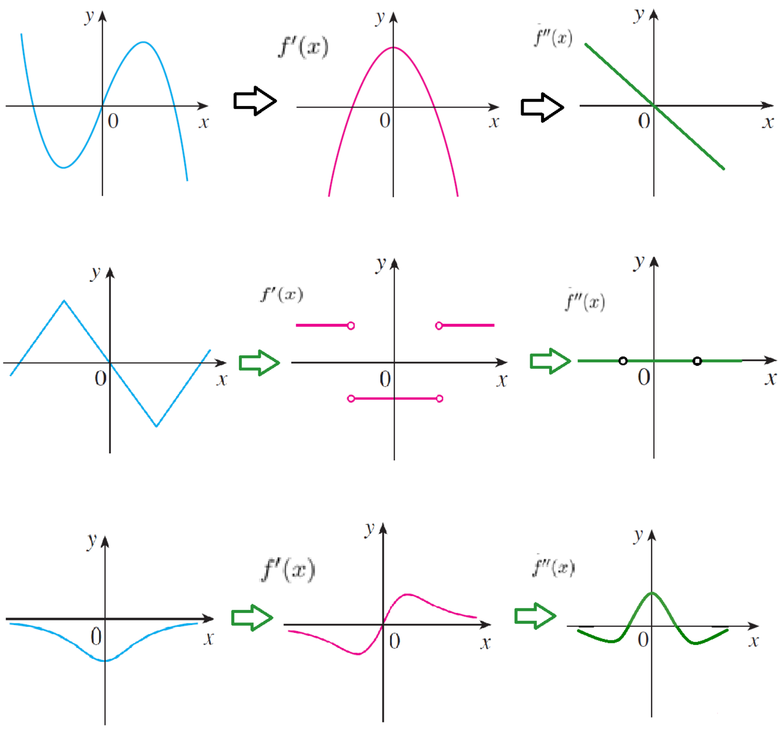
Problem 14

For each of the following graphs, copy the graph of the given function $f(x)$ onto your assignment sheet. (Assume that the axes have equal scales.) Then sketch the graphs of $f'(x)$ and $f''(x)$ directly below it.



(a)

(b)



(c)

(d)

Problem 15

Open the following link and

ll out the midterm course evaluation survey for this course. This survey is totally anonymous. Your feedback is very valuable to me as this will help me improve this course and teach you better for the rest of the semester. So please try to be as honest as possible with your responses. If you dislike something about the course, you are encouraged to pour your heart out in the comments.

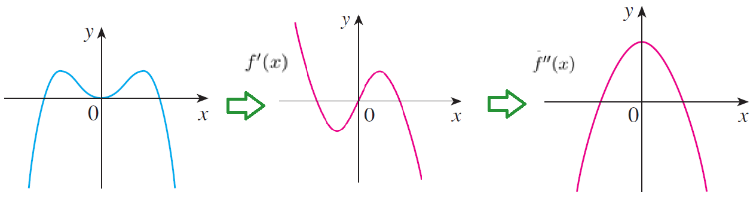
https://goo.gl/forms/Pdw7zQBNsLz3FmhB2

Answer the following questions while filling out the survey and write down the answers on your assignment.

(a) What is the 12th question in the survey?

(b) How many total questions are there in the survey?

Complete the survey and write down on your assignment "I have submitted the survey".
Solution is not required.


y
0
x
f′(x)
y
0
x
f″(x)
y
0
x