MT110: Calculus and Analytic Geometry

Homework 2 Solution

Due: Fri, Oct 12, 2:00 pm *Fall 2018*

Problem 1

(a) Let $f(x) = \dfrac{x-1}{\sqrt{x}-1}$

(i) Evaluate $f(1.5), f(1.1), f(1.01)$ and $f(1.001)$ and guess the value for $\lim_{x\to 1^+} f(x)$.

(ii) Evaluate $f(0.5), f(0.9), f(0.99)$ and $f(0.999)$ and guess the value for $\lim_{x\to 1^-} f(x)$.

(iii) Do you think $\lim_{x\to 1} f(x)$ exists?

Solution:

x	$f(x)$
1.5	2.224744871
1.1	2.048808848
1.01	2.004987562
1.001	2.000499875

$$\implies \lim_{x\to 1^+} f(x) = 2$$

x	$f(x)$
0.5	1.707106781
0.9	1.948683298
0.99	1.994987437
0.999	1.999499875

$$\implies \lim_{x\to 1^-} f(x) = 2$$

$$\implies \lim_{x\to 1} f(x) = 2$$

(b) Let $f(x) = \dfrac{x+1}{x^2-1}$

(i) Evaluate $f(-1.5), f(-1.1), f(-1.01)$ and $f(-1.001)$ and guess the value for $\lim_{x\to -1^-} f(x)$.

(ii) Evaluate $f(-0.5), f(-0.9), f(-0.99)$ and $f(-0.999)$ and guess the value for $\lim_{x\to -1^+} f(x)$.

(iii) Do you think $\lim_{x\to -1} f(x)$ exists?

Solution:

x	$f(x)$
-1.5	-0.4
-1.1	-0.476190476
-1.01	-0.497512438
-1.001	-0.499750125

$$\implies \lim_{x\to -1^-} f(x) = -0.5$$

x	$f(x)$
-0.5	-0.666666667
-0.9	-0.526315789
-0.99	-0.502512563
-0.999	-0.500250125

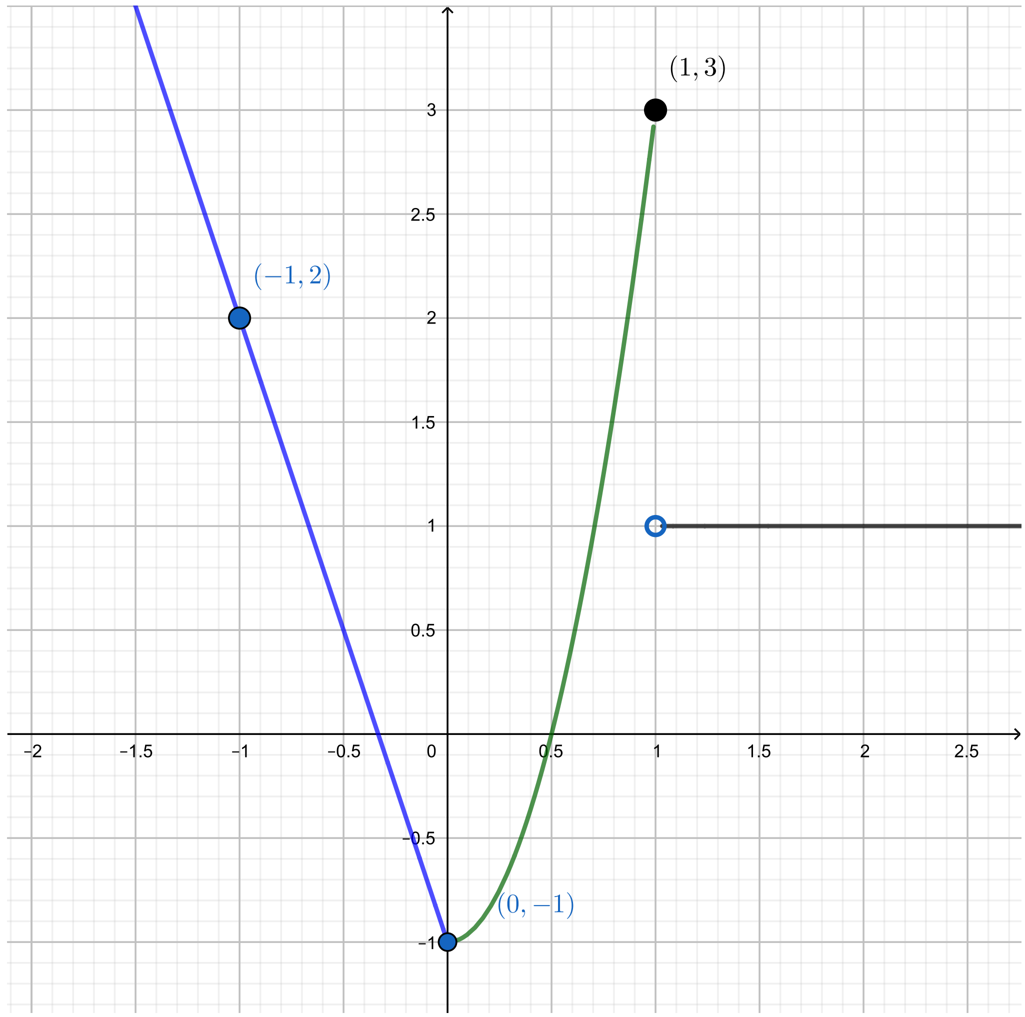
$$\implies \lim_{x\to -1^+} f(x) = -0.5$$

$$\implies \lim_{x\to -1} f(x) = -0.5$$

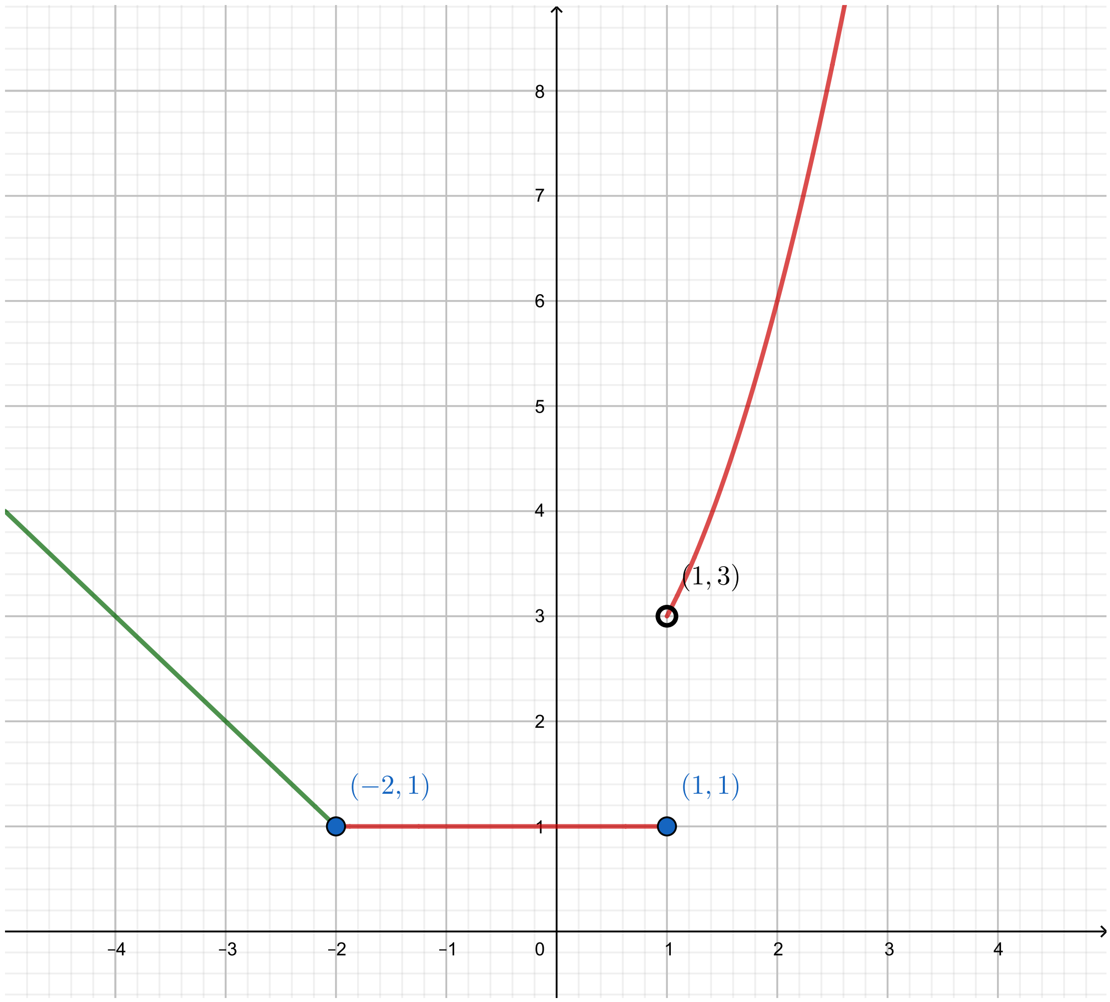
Problem 2

Sketch a possible graph of each of the following functions with the given properties.

(a) $f(-1) = 2, f(0) = -1, f(1) = 3$ and $\lim_{x \to 1} f(x)$ does not exist.

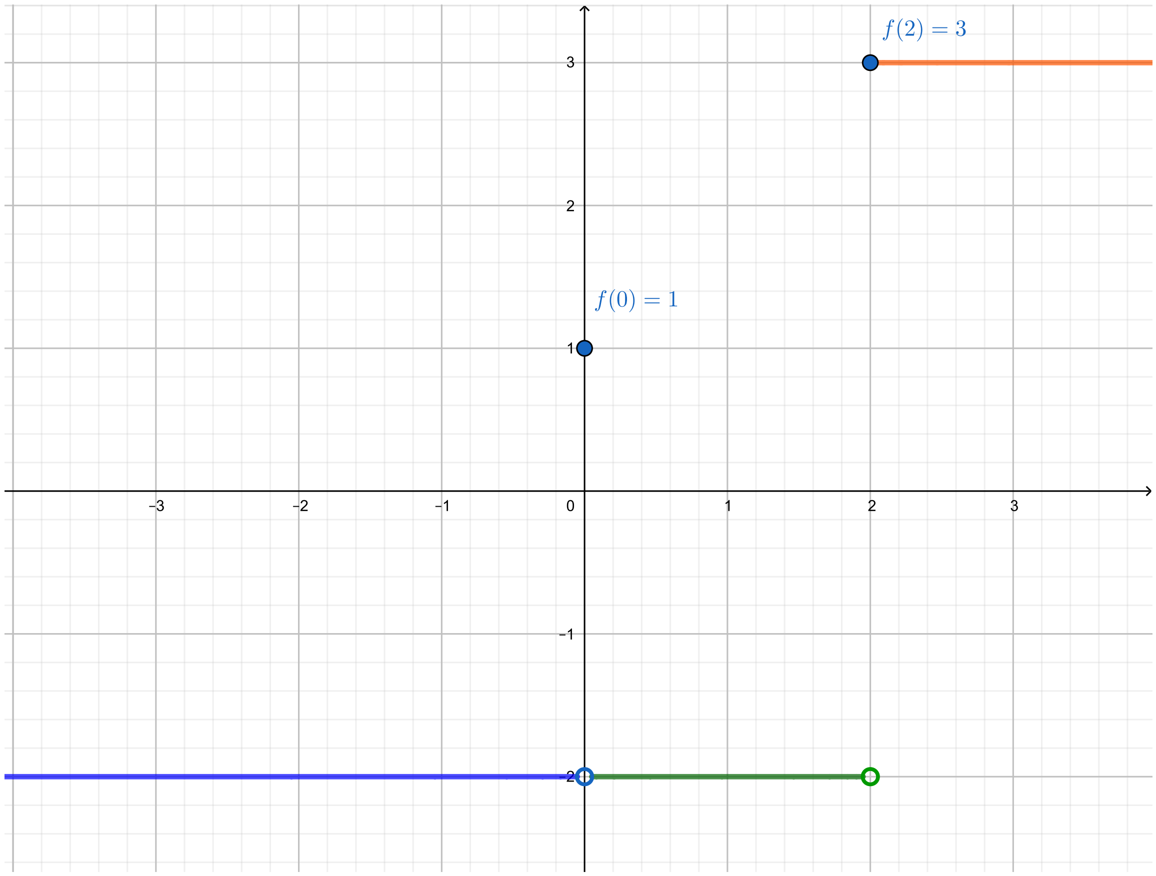


(b) $f(x) = 1$ for $-2 \leq x \leq 1$, $\lim_{x \to 1^+} f(x) = 3$ and $\lim_{x \to -2} f(x) = 1$.



(c) $f(0) = 1$, $\lim_{x \to 0^-} f(x) = 2$ and $\lim_{x \to 0^+} f(x) = 3$.

(d) $\lim_{x\to 0} f(x) = -2$, $f(0) = 1$, $f(2) = 3$ and $\lim_{x\to 2} f(x)$ does not exist.



Problem 3

(a) Numerically estimate $\lim_{x\to 0^+} x^{\sec x}$

(b) Explain what is wrong with the following logic:

$$\text{Since 0 to any power is 0, } \lim_{x\to 0} x^{\sec x} = \lim_{x\to 0} 0^{\sec x} = 0.$$

Solution:

x	$g(x) = x^{\sec(x)}$
0.1	0.0988505
0.05	0.0498129
0.005	0.0049997
0.0005	0.0005000
0.00005	0.0000500
0.000005	0.0000050
0.0000005	0.0000005
0.0000001	0.0000001

$$\implies \lim_{x\to 0^+} g(x) = 0$$

(b) Since, we are not evaluating at $x = 0$ but we are approaching to $x = 0$ Also, if you fixed the base while varying the exponent which wrong. In actual function both are changing.

Problem 4

(a) Give a possible expression of a function $f(x)$ such that $\lim_{x\to 0} f(x)$ exists but $f(0)$ does not exist.

(b) Give a possible expression of a function $g(x)$ such that $g(0)$ exists but $\lim_{x\to 0} g(x)$ does not exist.

(c) Give a possible expression of a function $f(x)$ such that $f(0)$ exists and $\lim_{x\to 0} f(x)$ exists but $\lim_{x\to 0} f(x) \neq f(0)$.

Solution:

(a) $f(x) = \dfrac{\sin(x)}{x}$

(b) $g(x) = \begin{cases} 1 + x^2, & \text{if } x \leq 0 \\ 2 + x, & \text{if } x > 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x \neq 0 \\ 6, & \text{if } x = 0 \end{cases}$

Problem 5

Find the exact value of the following limits. State clearly any limit rules or theorems that you use.

Solution:

(a) $\lim_{x\to 2} \dfrac{x+2}{x^2+5x+6}$

$= \dfrac{2+2}{2^2+5(2)+6} = \dfrac{1}{5} \quad \because$ Rational function

(b) $\lim_{x\to -1} \dfrac{x^2-1}{x+1}$

$\lim_{x\to -1} \dfrac{x^2-1}{x+1} = \lim_{x\to -1} \dfrac{(x-1)(x+1)}{x+1} = \lim_{x\to -1} (x-1) \quad \because x \neq -1$

$\implies \lim_{x\to -1} (x-1) = -2 \quad \because$ Polynomial

(c) $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$ [Hint: Rationalize]

$\lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2} = \lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2} \left(\dfrac{\sqrt{x+3}+2}{\sqrt{x+3}+2} \right) = \lim_{x\to 1} \dfrac{(x-1)(\sqrt{x+3}+2)}{x-1} = \lim_{x\to 1} \sqrt{x+3}+2 \because x \neq 1$

$\implies \lim_{x\to 1} \sqrt{x+3}+2 = 4 \quad \because$ Root function

(d) $\lim_{x\to 1} \dfrac{x^2-1}{x+2}$

$\lim_{x\to 1} \dfrac{x^2-1}{x+2} = 0 \quad \because$ Rational function

(e) $\lim_{x\to -1} \dfrac{x^2+x}{x^2-x-2}$

$\lim_{x\to -1} \dfrac{x(x+1)}{(x-2)(x+1)} = \lim_{x\to -1} \dfrac{x}{x-2} \quad \because x \neq -1$

$= \lim_{x\to -1} \dfrac{x}{x-2} = \dfrac{1}{3} \quad \because$ Rational function

(f) $\lim_{x\to 2} \frac{x^2-2x}{x^2-4}$

$\lim_{x\to 2} \frac{x^2-2x}{x^2-4} = \lim_{x\to 2} \frac{(x-2)x}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x}{x+2} \quad \because x \neq 2$

$\lim_{x\to 2} \frac{x}{x+2} = \frac{1}{2} \quad \because$ Rational function

(g) $\lim_{x\to \pi} x \sin x$

$(\lim_{x\to \pi} x)(\lim_{x\to \pi} \sin x) = (\pi)(0) = 0$

(h) $\lim_{x\to 0} \frac{\tan x}{\sin x}$

$\lim_{x\to 0} \frac{\tan x}{\sin x} = \lim_{x\to 0} \frac{\sin x}{(\cos x)(\sin x)} = \lim_{x\to 0} \frac{1}{\cos x} \quad \because \sin x \neq 0 \implies x \neq 0$

$\implies \lim_{x\to 0} \frac{1}{\cos x} = 1$

(i) $\lim_{x\to 2} \sqrt{x^2-2x+2}$

$\lim_{x\to 2} \sqrt{x^2-2x+2} = \sqrt{\lim_{x\to 2}(x^2-2x+2)} = \sqrt{2} \quad \because$ polynomial and root function

(j) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{1-\cos x}{x^2} \left(\frac{1+\cos x}{1+\cos x}\right)$ for $\cos x \neq -1$

$\lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \lim_{x\to 0} \frac{1}{1+\cos x} = (1)^2 \left(\frac{1}{2}\right) = \frac{1}{2}$

(k) $\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right)$

We know that for $x \neq 0$,

$$-1 \leq \cos\left(\frac{2}{x}\right) \leq 1$$

$$-x^4 \leq x^4 \cos\left(\frac{2}{x}\right) \leq x^4$$

$$\lim_{x\to 0}(-x^4) \leq \lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) \leq \lim_{x\to 0} x^4$$

$$0 \leq \lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) \leq 0$$

So the only possible value of this limit is 0.

$$\implies \lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$$

This is called sandwich theorem.

Problem 6

(a) Write each answer as an equality, and any decimals up to 7 decimal places (e.g. $2 < x < 2.0164389$). Clearly state the value of ϵ and δ (or M or N and δ) in each case.

(i) How close to 4 do we need to take x so that $\left(\frac{x}{2} - 2\right) < 0.001$?

$$\left(\frac{x-4}{2}\right) < 0.001$$
$$\implies (x-4) < 0.002$$
$$\implies x < 4 + 0.002 = 4.0020000$$
$$4 < x < 4.0020000$$

(ii) How close to 4 do we need to take x so that $\left(\frac{x}{2} - 2\right) > -0.0001$?

$$\left(\frac{x-4}{2}\right) > -0.0001$$
$$\implies (x-4) > -0.0002$$
$$\implies x > -0.0002 + 4 = 3.9998000$$
$$3.9998000 < x < 4$$

(iii) How close to 0 do we need to take x so that $(2x+9) < 9.0001$?

$$(2x+9) < 9.0001$$
$$\implies x < \frac{9.0001 - 9}{2} = 0.0000500$$
$$0 < x < 0.0000500$$

(iv) How close to 0 do we need to take x so that $(2x+9) > 8.999$?

$$(2x+9) > 8.999$$
$$x > \frac{8.999 - 9}{2} = -0.0005000$$
$$-0.0005000 < x < 0$$

(v) How close to 0 do we need to take x so that $\left(x^2 + 6x + 9\right) < 9.001$?

$$\left(x^2 + 6x + 9\right) < 9.001$$
$$\implies (x+3)^2 < 9.001$$
$$\implies -\sqrt{9.001} < x + 3 < \sqrt{9.001}$$
$$\implies -\sqrt{9.001} - 3 < x < \sqrt{9.001} - 3$$
$$-6.00017 < x < 0.000166662$$

Since we need x close to 0 so we will select positive square root quantity.

$$0 < x < 0.0001667$$

(vi) How close to 0 do we need to take x so that $\left(x^2+6x+9\right) < 9.0001$?

$$\left(x^2+6x+9\right) < 9.0001 \implies (x+3)^2 < 9.0001$$
$$\implies -\sqrt{9.0001} < x+3 < \sqrt{9.0001}$$
$$\implies -\sqrt{9.0001}-3 < x < \sqrt{9.0001}-3$$
$$-6.00002 < x < 0.0000166666$$

Since we need x close to 0 so we will select positive square root quantity.

$$0 < x < 0.0000167$$

(vii) How close to 0 do we need to take x so that $\left(x^2+6x+9\right) > 8.9999$?

$$\left(x^2+6x+9\right) > 8.9999$$
$$\implies (x+3)^2 > 8.9999$$
$$\implies \text{Either } x+3 > \sqrt{8.9999}$$
$$\implies x > \sqrt{8.9999}-3$$
$$x > -0.0000166667$$

Also,

$$\implies \text{Or } x+3 < -\sqrt{8.9999}$$
$$\implies x < -\sqrt{8.9999}-3$$
$$x < -5.99998$$

Since we need x close to 0 so we will select positive square root quantity.

$$-0.0000167 < x < 0$$

(viii) How close to -7 do we need to take x so that $\frac{1}{(x+7)^4} > 10000$?

$$\frac{1}{(x+7)^4} > 10000$$
$$(x+7)^4 < \frac{1}{10000}$$
$$-\sqrt[4]{\frac{1}{10000}} < (x+7) < \sqrt[4]{\frac{1}{10000}}$$
$$\implies -\sqrt[4]{\frac{1}{10000}}-7 < x < \sqrt[4]{\frac{1}{10000}}-7$$
$$\implies -7.1 < x < -6.9$$

(ix) How close to -7 do we need to take x so that $\frac{1}{(x+7)^4} > 100000$?

$$\frac{1}{(x+7)^4} > 100000$$
$$\implies (x+7)^4 < \frac{1}{100000}$$
$$\implies -\sqrt[4]{\frac{1}{100000}} < (x+7) < \sqrt[4]{\frac{1}{100000}}$$
$$\implies -\sqrt[4]{\frac{1}{100000}} - 7 < x < \sqrt[4]{\frac{1}{100000}} - 7$$
$$\implies -7.05 < x < -6.94$$

(x) How close to 0 do we need to take x so that $\ln x < -10000$?

$$\ln x < -10000$$
$$\implies e^{\ln x} < e^{-10000}$$
$$\implies x < e^{-10000}$$
$$0 < x < e^{-10000}$$

(xi) How close to 0 do we need to take x so that $\ln x < -100000$?

$$\ln x < -100000$$
$$\implies e^{\ln x} < e^{-100000}$$
$$\implies x < e^{-100000}$$
$$0 < x < e^{-100000}$$

(b) Use the ϵ-δ definition of a limit to show that

(i) $\lim_{x \to 4} \left(\frac{x}{2} - 2\right) = 0$

Let $\delta = 2\epsilon$. Starting from considering $|x - a| < \delta$.

$$|x-4| < \delta = 2\epsilon$$
$$\implies \left|\frac{x-4}{2}\right| < \epsilon$$
$$\left|\frac{x}{2} - 2\right| < \epsilon$$
$$\left|\left(\frac{x}{2} - 2\right) - 0\right| < \epsilon$$
$$\therefore |f(x) - L| < \epsilon$$

(ii) $\lim_{x \to 0} (2x + 9) = 9$

Let $\delta = \frac{\epsilon}{2}$. Starting from considering $|x - a| < \delta$

$$|x - 0| < \delta = \frac{\epsilon}{2}$$
$$\implies |2x| < \epsilon$$
$$|2x + 9 - 9| < \epsilon$$
$$|(2x + 9) - 9| < \epsilon$$
$$\therefore |f(x) - L| < \epsilon$$

(iii) $\lim_{x \to 0} (x^2 + 6x + 9) = 9$

Let $\delta = \min\left\{1, \frac{\epsilon}{7}\right\}$.($\because$ There may exist multiple deltas and minimum will be our desired delta.)

Starting from considering $|x - a| < \delta$.

$$|x - 0| < \delta = 1$$
$$\implies |x + 6| < 7$$

We also have

$$|x - 0| < \delta = \frac{\epsilon}{7}$$
$$\implies |x| < \frac{\epsilon}{7}$$

So

$$|x||x + 6| < (7)\left(\frac{\epsilon}{7}\right) = \epsilon$$
$$|x(x + 6)| < |x||x + 6| < \epsilon$$
$$\implies |x(x + 6)| < \epsilon$$
$$|x^2 + 6x + 9 - 9| < \epsilon$$
$$|(x^2 + 6x + 9) - 9| < \epsilon$$
$$\therefore |f(x) - L| < \epsilon$$

(iv) $\lim_{x \to -7} \frac{1}{(x + 7)^4} = \infty$

Let $\delta = \frac{1}{\sqrt[4]{M}}$. Starting from considering $|x - a| < \delta$

$$|x - (-7)| < \delta = \frac{1}{\sqrt[4]{M}}$$
$$|x + 7| < \frac{1}{\sqrt[4]{M}}$$
$$|x + 7|^4 < \frac{1}{M}$$
$$\left|\frac{1}{x + 7}\right|^4 > M$$
$$\frac{1}{(x + 7)^4} > M$$
$$\therefore f(x) > M$$

(v) $\lim_{x \to 0^+} \ln x = -\infty$

Let $\delta = e^N$. Starting from considering $|x - a| < \delta$.

$$|x - 0| < \delta = e^N$$

$$|x| < e^N$$

$$\ln|x| < \ln e^N = N$$

$$\ln|x| < \ln e^N = N$$

$$\ln|x| < N$$

$$\therefore f(x) < N$$