MT110: **Calculus and Analytic Geometry**

# Homework 1 Solution

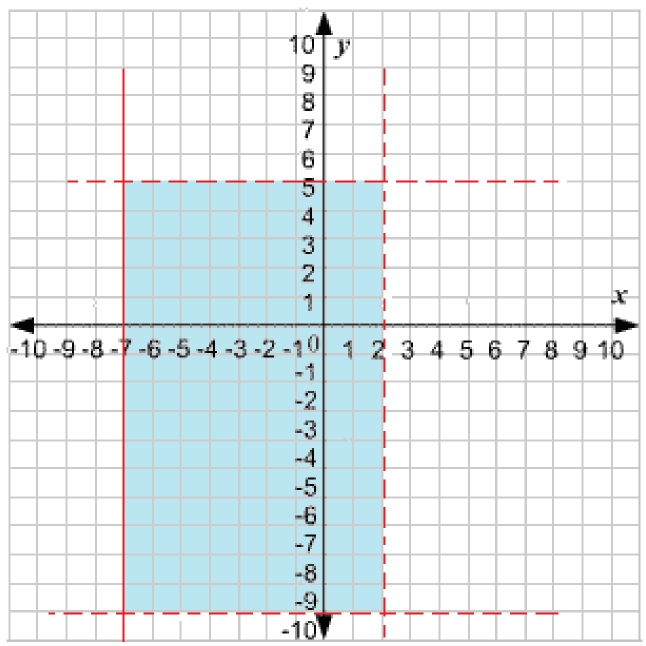
*Fall 2018*

## Problem 1

Draw the following regions in Cartesian plane.

(a) $R = \{x, y : -7 \leq x < 2 \text{ and } -10 < y < 5\}$

(a) Let's draw the boundaries first: solid line $x = -7$, and dotted lines $x = 2$, $y = -10$ and $y = 5$. As $x < 2$, $y > -10$ and $y < 5$, shade the region inside the rectangle.



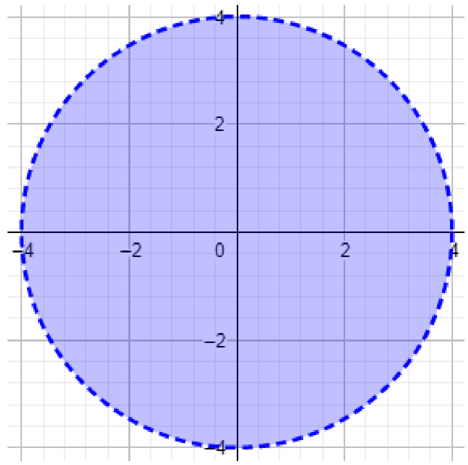

To verify whether the required shaded region is the one inside the boundaries and not outside, take a point inside the boundaries, let's say $(0, 0)$. Because $-7 \leq 0 < 2$ and $-10 < 0 < 5$, we have shaded the correct region.

(b) $R = \{x, y : 2x^2 + 2y^2 < 32\}$

Let's consider the boundary first.

$$2x^2 + 2y^2 = 32$$
$$x^2 + y^2 = 16$$

is an equation of the circle of radius 4. Draw the circle and shade the region inside the circle because the required region is $x^2 + y^2 < 16$.




To verify, consider a point in our shaded region let's say $(1, 1)$. Then $1^2 + 1^2 = 2 < 16$. So, required shaded region is inside the circle.

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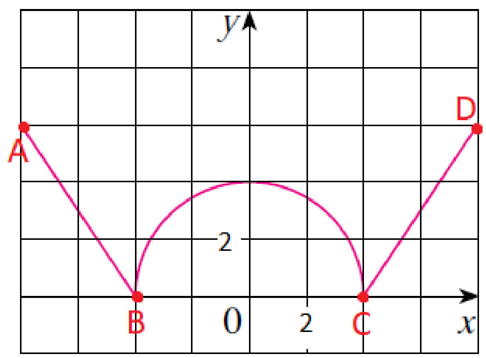
## Problem 2

Find an expression for the function whose graph is the given curve.




The function can be divided into three pieces and their corresponding intervals in its domain are

$-8 \leq x < -4$, $-4 \leq x \leq 4$, and $4 < x \leq 8$

(a) Piece 1: Straight line passes through two points A$(-8, 6)$ and B$(-4, 0)$. Using $y = mx + c$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 6}{-4 - (-8)} = -\frac{3}{2}$$

$$c = y - mx = 0 - \left(-\frac{3}{2}\right)(-4) = -6$$

$$y = -\frac{3}{2}x - 6 \qquad \text{for} \quad -8 \leq x < -4$$

(b) Piece 2: is the upper ($y > 0$) semi-circle of radius 4.

$$y^2 + x^2 = 4^2$$

$$y = \sqrt{16 - x^2} \qquad \text{for} \quad -4 \leq x \leq 4$$

(c) Piece 3: Straight line passes through two points C(4, 0) and D(8, 6).

$$m = \frac{3}{2}$$

$$c = -6$$

$$y = \frac{3}{2}x - 6 \qquad \text{for} \quad 4 < x \leq 8$$

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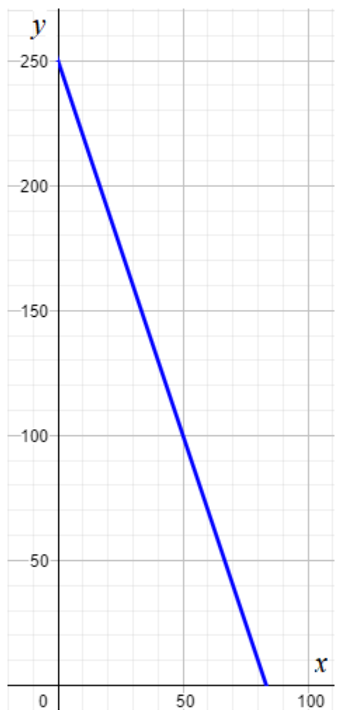
## Problem 3

The manager of a weekend flea market knows from past experience that if he charges $x$ dollars for a rental space at the market, then the number of spaces $y$ he can rent is given by the equation $y = 250 - 3x$.

(a) Sketch a graph of this linear function. (Remember that the rental charge per space and the number of spaces rented cannot be negative quantities.)




(b) What do the slope, the $x$-intercept, and the $y$-intercept of the graph represent in the actual scenario?

It is important to note than $x$, the rent of a space, in the independent variables and $y$, number of rental spaces, is the dependent variable.

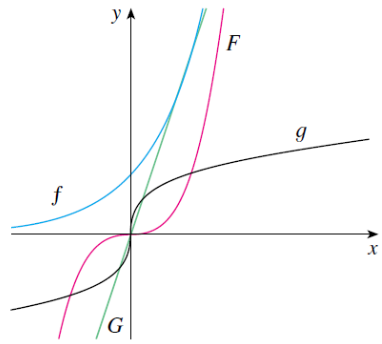
Slope: $m = \frac{y_2 - y_1}{x_2 - x_1}$. It represents the ratio of change in number of spaces to the change in the rent of a space. A slope of $-3$ represent that if the charges of rental space in the market increase by \$1, the number of spaces the manager can rent decrease by 3. It also means that if the charges of rental space in the market decrease by \$1, the number of spaces the manager can rent increase by 3.

$x$-intercept ($x = 83.33, y = 0$): If he charges \$83.33 for a space, there will be no spaces in the market that he can rent.

$y$-intercept ($x = 0, y = 250$): If he gives away the spaces for free, there will be a total of 250 spaces that he can give away.

## Problem 4

Match each equation with its graph. Explain your choices.




(a) $y = 3x$ is $G$

(b) $y = 3^x$ is $f$

(c) $y = x^3$ is $F$

(d) $y = \sqrt[3]{x}$ is $g$

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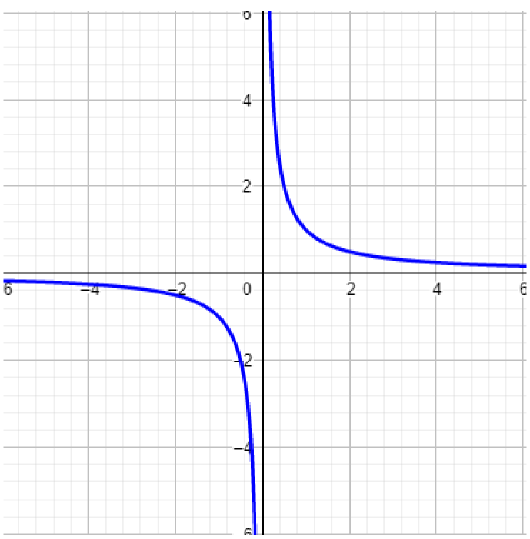
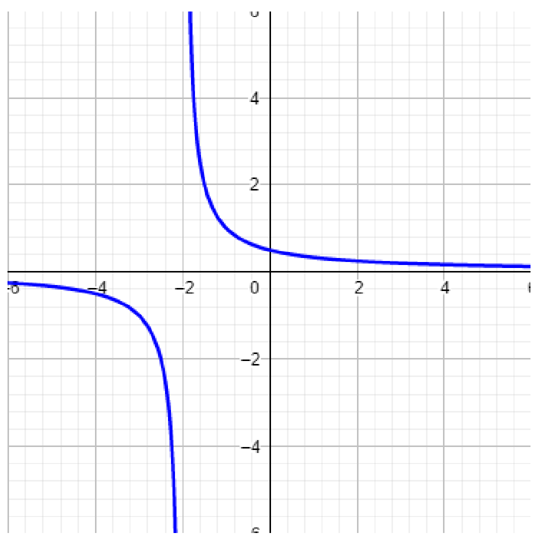
## Problem 5

Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations.

(a) $y = \dfrac{1}{x+2}$

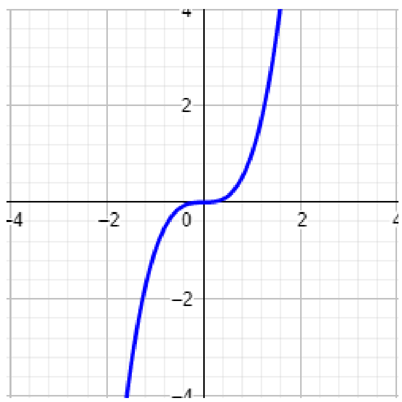



$y = \dfrac{1}{x}$

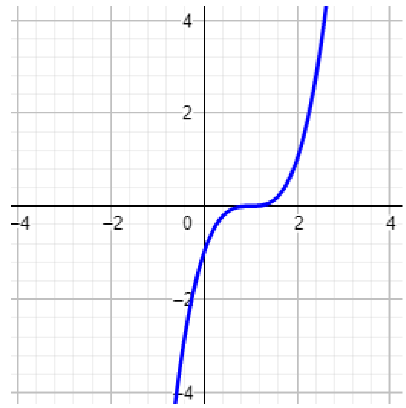



$y = \dfrac{1}{x+2}$

(b) $y = (x-1)^3$

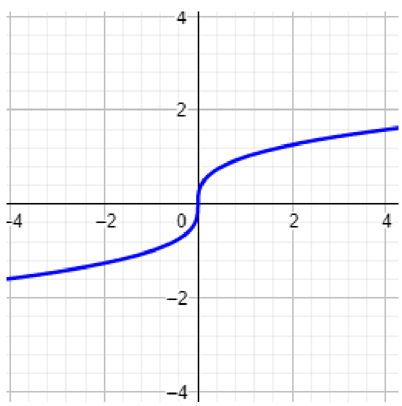


$f(x) = x^3$

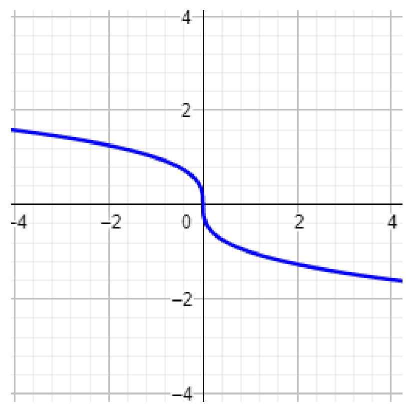


$y = (x-1)^3$

(c) $y = -\sqrt[3]{x}$



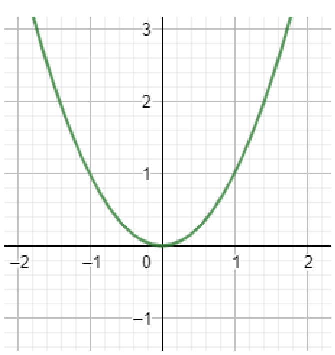
$f(x) = \sqrt[3]{x}$



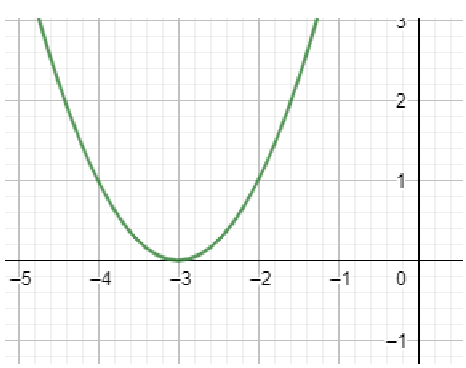
$f(x) = -\sqrt[3]{x}$

(d) $y = x^2 + 6x + 4$

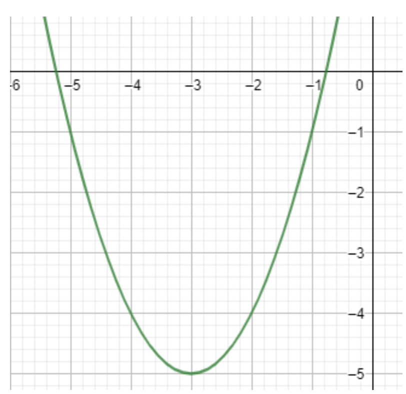
$$y = x^2 + 6x + 4 = x^2 + 6x + 9 - 9 + 4$$
$$\implies y = (x+3)^2 - 5$$



$f(x) = x^2$

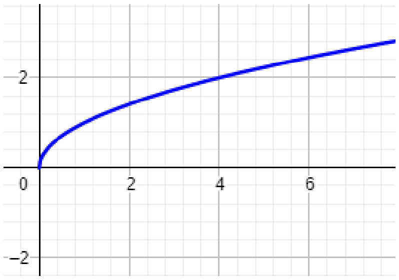


$f(x) = (x+3)^2$

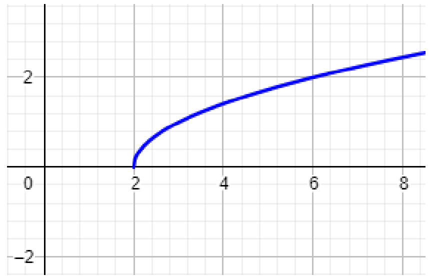


$f(x) = (x+3)^2 - 5$

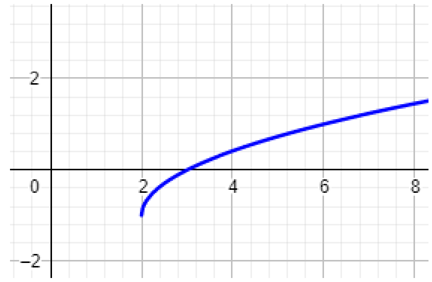
(e) $y = \sqrt{x-2} - 1$




$y = \sqrt{x}$

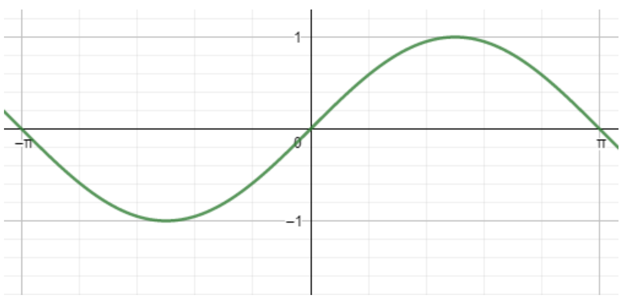



$y = \sqrt{x-2}$

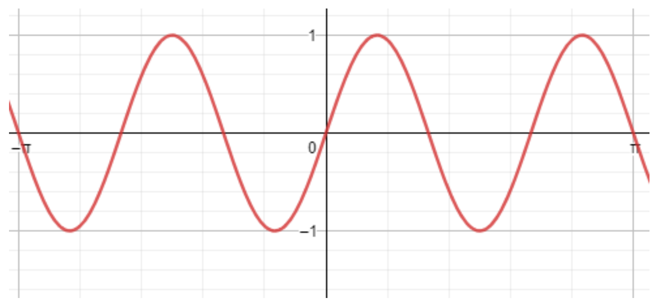



$y = \sqrt{x-2} - 1$

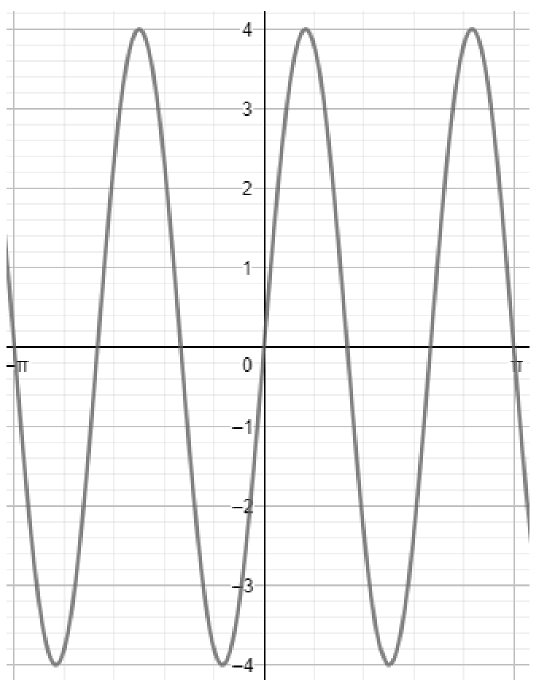
(f) $y = 4\sin 3x$




$y = \sin x$

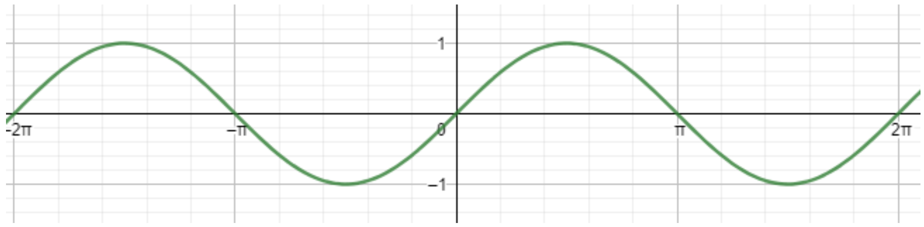



$y = \sin 3x$

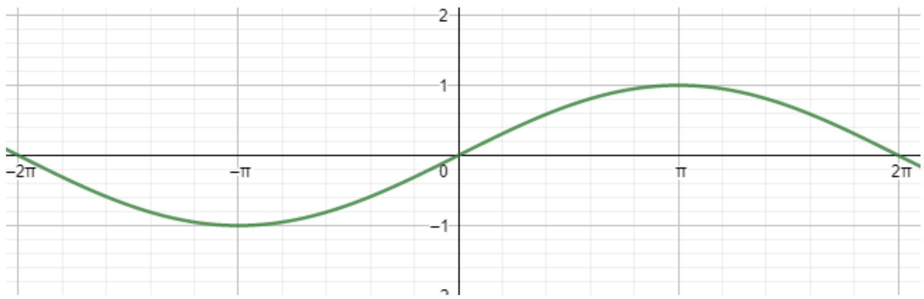



$y = 4\sin 3x$

(g) $y = \sin\left(\frac{1}{2}x\right)$

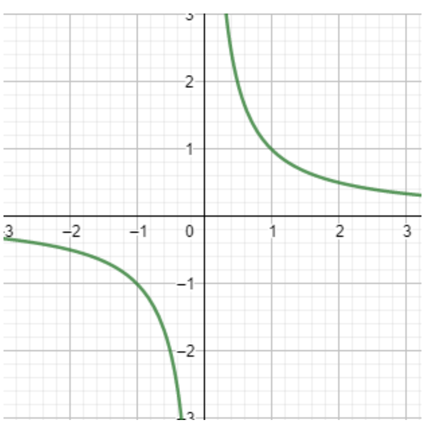



$y = \sin x$

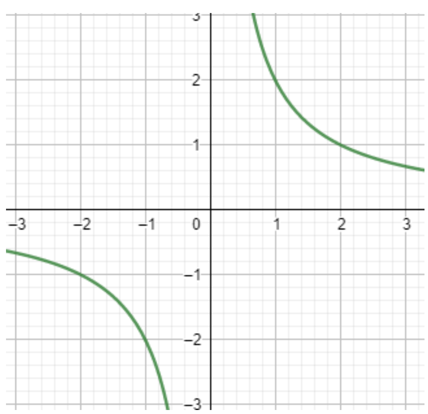



$y = \sin\left(\frac{1}{2}x\right)$

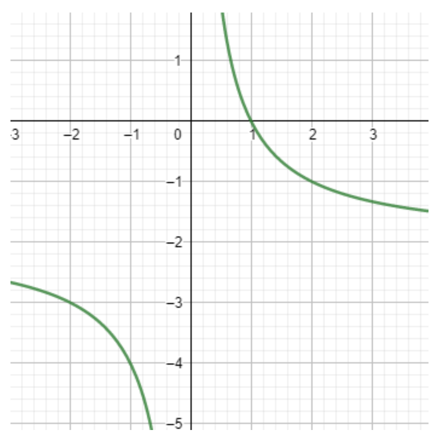
(h) $y = \dfrac{2}{x} - 2$



$y = \dfrac{1}{x}$

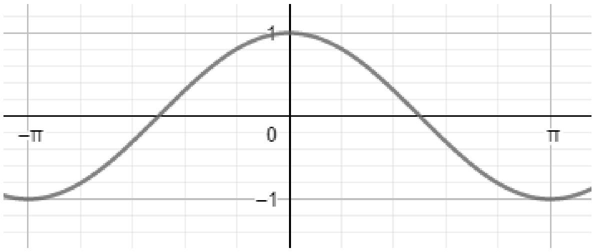


$y = \dfrac{2}{x}$

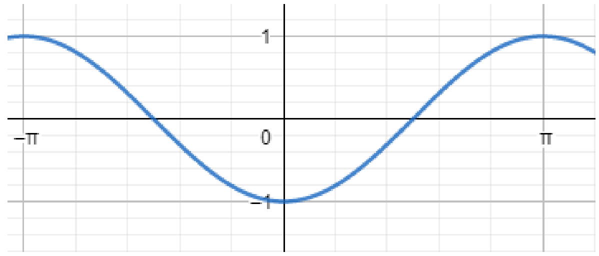


$y = \dfrac{2}{x} - 2$

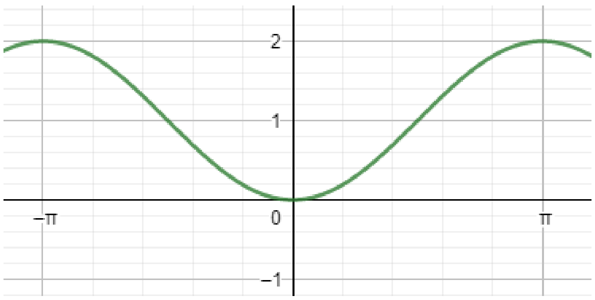
(i) $y = \dfrac{1}{2}(1 - \cos x)$



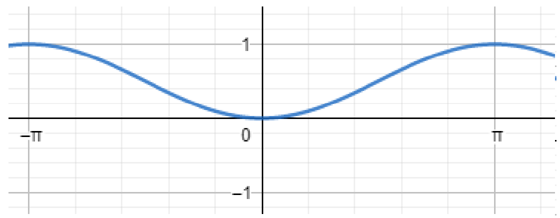

$y = \cos x$




$y = -\cos x$

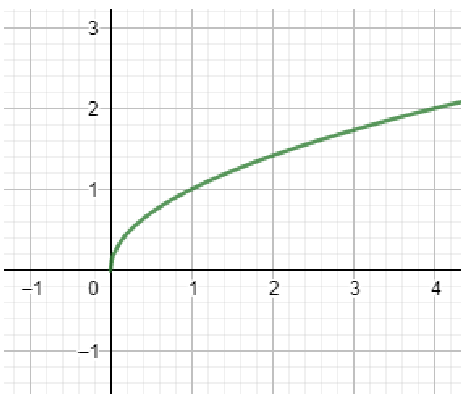



$y = -\cos x + 1$

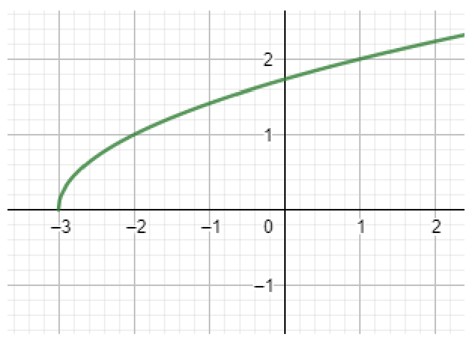



$y = \frac{1}{2}(1 - \cos x)$

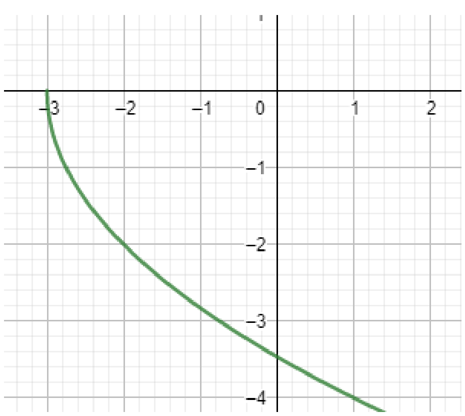
(j) $y = 1 - 2\sqrt{x + 3}$



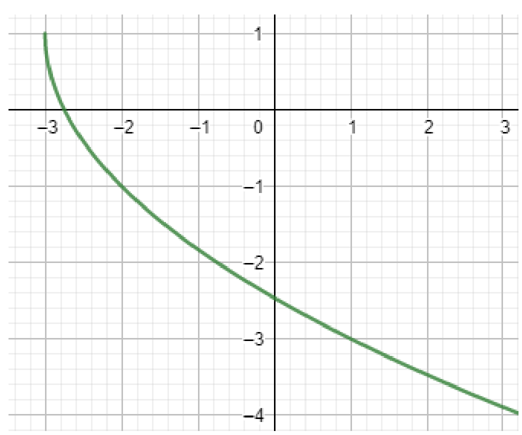

$y = \sqrt{x}$




$y = \sqrt{x + 3}$

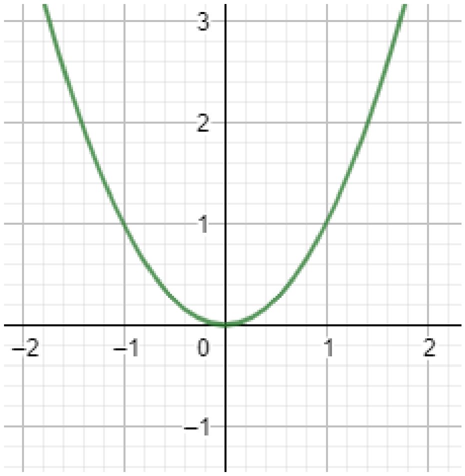



$y = -2\sqrt{x + 3}$

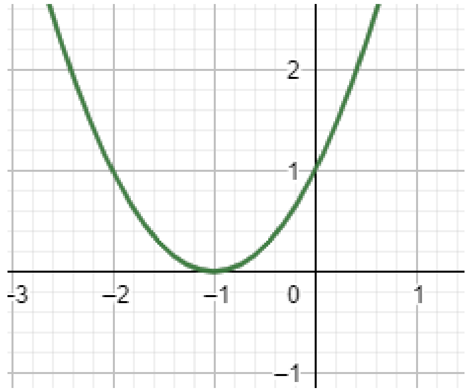



$y = 1 - 2\sqrt{x + 3}$

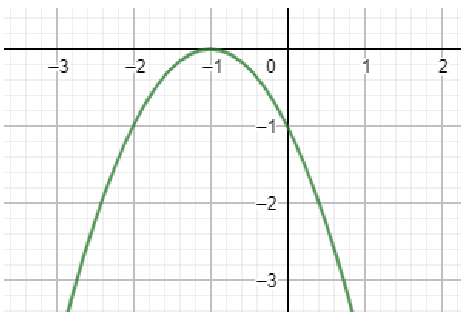
(k) $y = 1 - 2x - x^2$



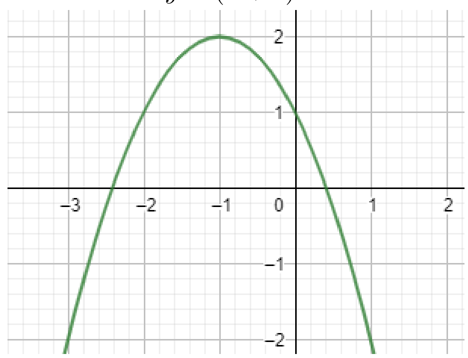
$y = x^2$



$y = (x+1)^2$

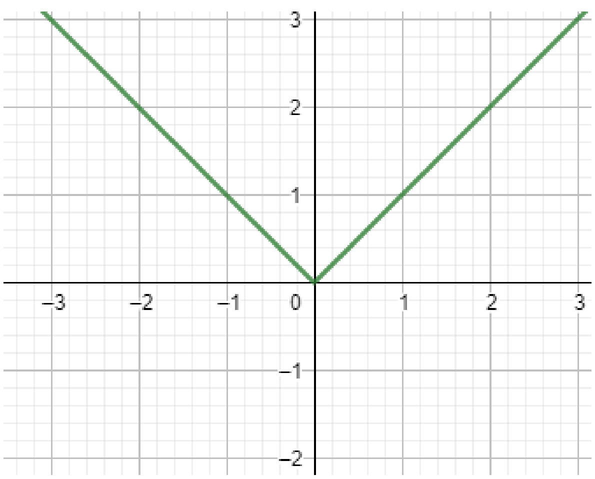


$y = -(x+1)^2$

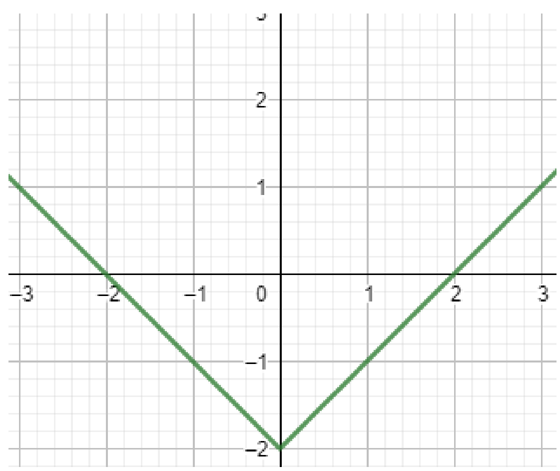


$y = -(x+1)^2 + 2$

(l) $y = |x| - 2$

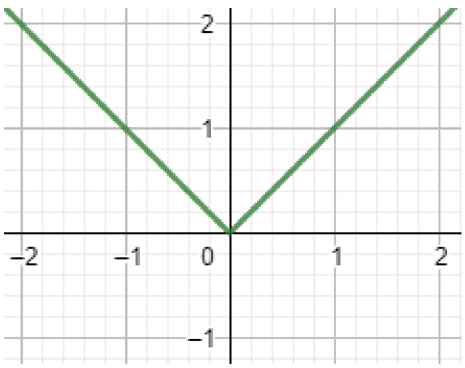


$y = |x|$

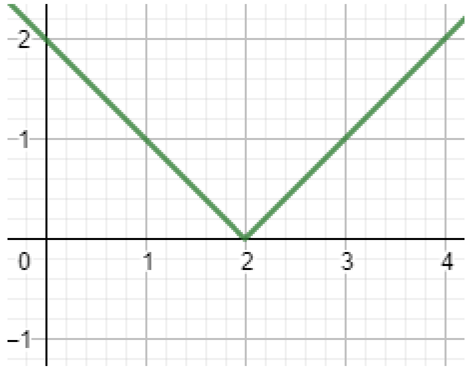


$y = |x| - 2$

(m) $y = |x - 2|$

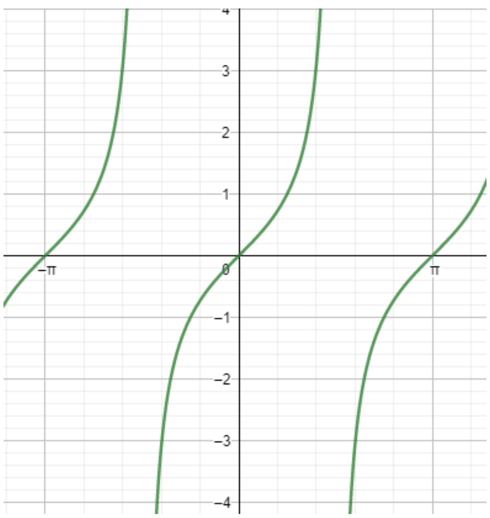



$y = |x|$

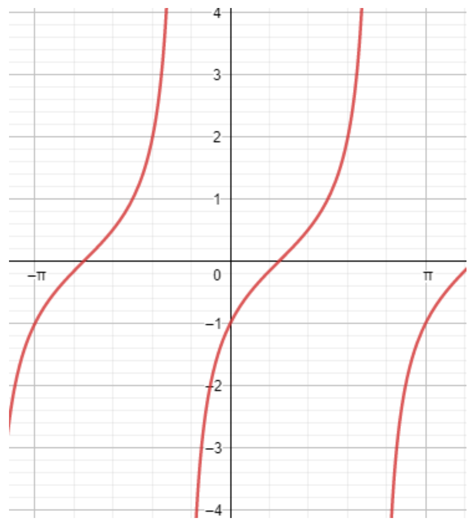



$y = |x - 2|$

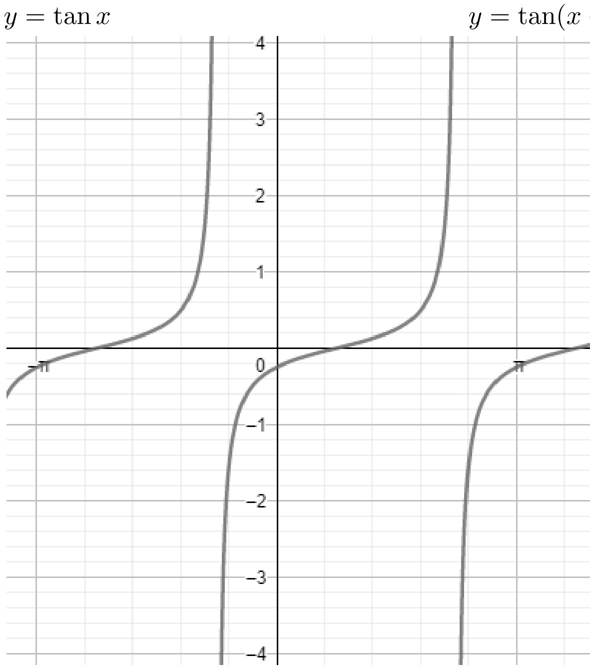
(n) $y = \frac{1}{4} \tan\left(x - \frac{\pi}{4}\right)$




$y = \tan x$




$y = \tan(x - \frac{\pi}{4})$



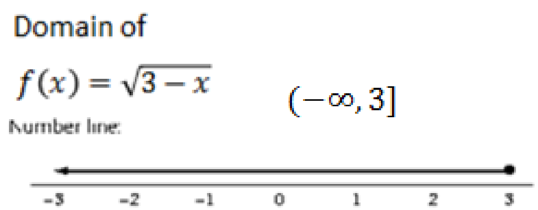

(a) $y = \frac{1}{4} \tan\left(x - \frac{\pi}{4}\right)$

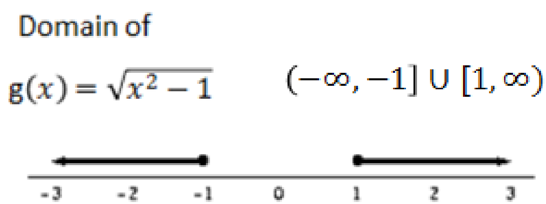
## Problem 6

Given $f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{x^2-1}$. Find the following combinations of functions and state the domain in each case.

(a) $f+g$ (b) $f-g$ (c) $fg$ (d) $f/g$

$$D_f = (-\infty, 3] \text{ and } D_g = (-\infty, -1] \cup [1, \infty)$$




(a) $(f+g)(x) = \sqrt{3-x} + \sqrt{x^2-1}$

$$D_{f+g} = D_f \cap D_g = (-\infty, -1] \cup [1, 3]$$

(b) $(f-g)(x) = \sqrt{3-x} - \sqrt{x^2-1}$

$$D_{f-g} = D_f \cap D_g = (-\infty, -1] \cup [1, 3]$$

(c) $(fg)(x) = \sqrt{(3-x)(x^2-1)}$

$$D_{fg} = D_f \cap D_g = (-\infty, -1] \cup [1, 3]$$

(d) $(f/g)(x) = \sqrt{\dfrac{3-x}{x^2-1}}$

$$D_{f/g} = (D_f \cap D_g) \backslash \{\pm 1\} = (-\infty, -1) \cup (1, 3]$$

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## Problem 7

Let $f(x) = x + \dfrac{1}{x}$ and $g(x) = \dfrac{x+1}{x+2}$, find the following composite functions and their domains.

(a) $f \circ g$

(b) $g \circ f$

(c) $f \circ f$

(d) $g \circ g$

The domain of $f$, $D_f = (-\infty, 0) \cup (0, \infty)$. We can verify this by finding the domains and ranges of $x$ and $\frac{1}{x}$ separately and taking their intersection, using the same method as done in Problem 6.

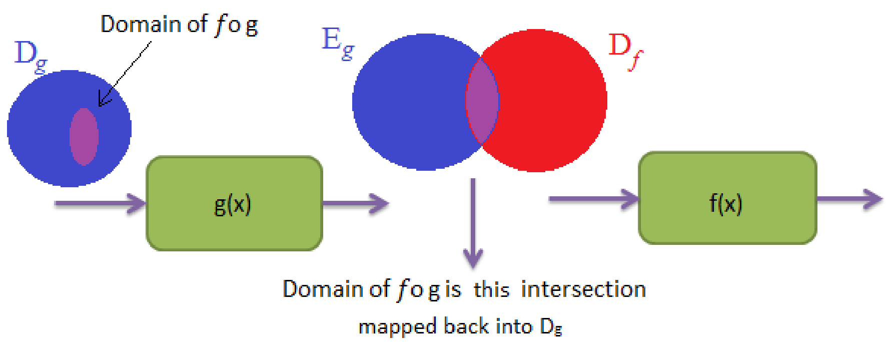
The range of $f$, $E_f = (-\infty, 0) \cup (0, \infty)$, which is equal to the domain of $f^{-1}$.

Similarly $D_g = (-\infty, -2) \cup (-2, \infty)$ and $E_g = \left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$.

(a) $f \circ g = f(g(x)) = f\left(\frac{x+1}{x+2}\right) = \dfrac{x+1}{x+2} + \dfrac{1}{\frac{x+1}{x+2}} = \dfrac{x+1}{x+2} + \dfrac{x+2}{x+1} \implies x \neq -2$ and $x \neq -1$

$\implies D_{f \circ g} = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

This domain can also be found using the method found in the following figure.




(b) $g \circ f = g(f(x)) = g\left(x + \frac{1}{x}\right) = \dfrac{x + \frac{1}{x} + 1}{x + \frac{1}{x} + 2} = \dfrac{\frac{x^2+1+x}{x}}{\frac{x^2+1+2x}{x}} = \dfrac{x(x^2+x+1)}{x(x^2+2x+1)} = \dfrac{x(x^2+x+1)}{x(x+1)^2}$

$\implies x \neq 0$ and $x \neq -1$. Notice that we cannot cancel $x$ in the numerator and the denominator because $x = 0$ is allowed in the domain of $g$. If we cancel, we will also miss the condition that $x \neq 0$. Because $x = 0$ is not in the domain of $f$ and hence it cannot be in the domain of $g \circ f$ either.

$\implies D_{g \circ f} = (-\infty, -1) \cup (-1, 0) \cup (0, \infty)$.

(c) $f \circ f = f(f(x))x = f\left(x + \frac{1}{x}\right) = x + \frac{1}{x} + \dfrac{1}{x + \frac{1}{x}} = x + \frac{1}{x} + \dfrac{x}{x^2+1} = \dfrac{x^4 + 3x^2 + 1}{x(x^2+1)}$

$\implies x \neq 0$

$\implies D_{f \circ f} = (-\infty, 0) \cup (0, \infty)$.

(d) $g \circ g = g(g(x)) = g\left(\frac{x+1}{x+2}\right) = \dfrac{\frac{x+1}{x+2} + 1}{\frac{x+1}{x+2} + 2} = \dfrac{\frac{2x+3}{x+2}}{\frac{3x+5}{x+2}} = \dfrac{(2x+3)(x+2)}{(3x+5)(x+2)}$

$\implies x \neq -\frac{5}{3}$ and $x \neq -2$.

$\implies D_{g \circ g} = (-\infty, -2) \cup \left(-2, -\frac{5}{3}\right) \cup \left(-\frac{5}{3}, \infty\right)$.

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## Problem 8

(a) Suppose $f$ is a one-to-one function with domain D and range E. How is the inverse function $f^{-1}$ defined? What is the domain of $f^{-1}$? What is the range of $f^{-1}$?

Suppose $f$ is a function that takes every element $x \in$ D and returns the elements $y \in$ E. Its inverse $f^{-1}$ is a function that takes every element $y$ in the range E of $f$ and returns the corresponding elements $x$ in the domain D of $f$. Therefore the domain of $f^{-1}$ is E and the range of $f^{-1}$ is D.

(b) If you are given a formula for $f$, how do you find a formula for $f^{-1}$?

Steps to find formula of $f^{-1}$:

(1) Write $f(x) = y$

(2) Solve this equation for $x$ in terms of $y$ ( if possible).

(3) To express $f^{-1}$ as a function of $x$, interchange $x$ and $y$. The resulting equation is $y = f^{-1}(x)$.

(c) If you are given the graph of $f$, how do you find the graph of $f^{-1}$?

If graph of $f(x)$ is given then graph of $f^{-1}$ is obtained by reflecting the graph of $f$ in the line $y = x$.

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## Problem 9

Let $f(x) = \sqrt{1 - x^2}, 0 \leq x \leq 1$.

(a) Find $f^{-1}(x)$. How is it related to $f(x)$?

Step 1: Solve this equation for $x$ in terms of $y$ ( if possible).

$$y = \sqrt{1 - x^2}$$
$$y^2 = 1 - x^2$$
$$x^2 = 1 - y^2$$
$$x = \sqrt{1 - y^2}$$

We chose postive square root because we know that $0 \leq x \leq 1$.

Step 2: Interchange $x$ and $y$.

$$y = \sqrt{1 - x^2}$$

Step 3: $f^{-1}(x) = \sqrt{1 - x^2}$

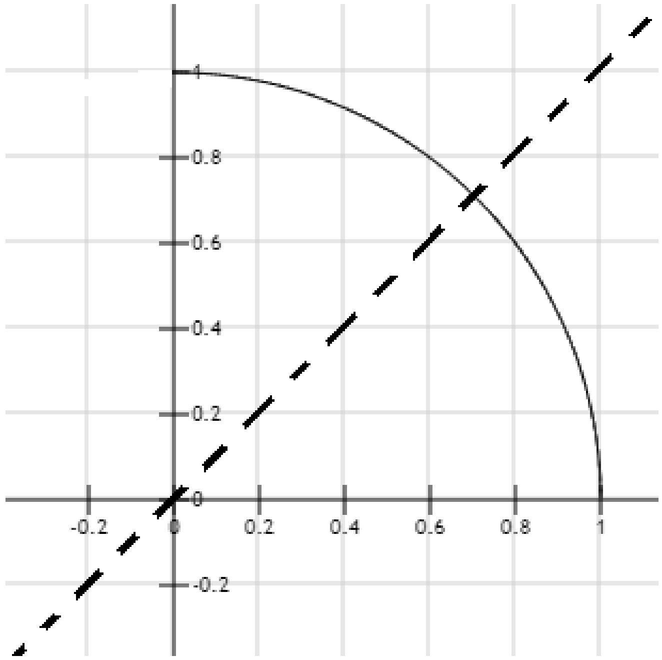
The result shows that $f^{-1}(x) = f(x)$ because

(1) They have the same expressions.

(2) Their respective domain and range are identical.

(b) Sketch the graph of $f$ and explain your answer to part (a).

Because $f(x) = f^{-1}(x)$, it can be seen from the graph that the function is symmetric across the line $y = x$.

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## Problem 10

We know that domain of function $f(x) = x^2$ is all real numbers because we can square all real numbers. Why is it true that its range consists of all positive real numbers, without leaving out any numbers in between? [Hint: Use the concept of an inverse function]

Coming soon ... wait for it ... it's going to be very pretty.

## Problem 11

For each of the following functions,

(a) $f(x) = \dfrac{4x-1}{2x+3}$

(b) $f(x) = \ln(x+3)$

(i) Find its domain and range

(ii) Find a formula for the inverse of the function and find its domain and range.

(a) (i) $f(x) = \dfrac{4x-1}{2x+3}$

$D_f = \left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, \infty\right)$.

To find the range $E_f$, let's find the inverse function first. Because the domain of $f^{-1}$ will be the range of $f$.

(ii)

$$\text{Let } y = \frac{4x-1}{2x+3}$$
$$(2x+3)y = 4x-1$$
$$2xy + 3y = 4x - 1$$
$$3y + 1 = x(4-2y)$$
$$x = \frac{3y+1}{4-2y}$$
$$\implies f^{-1}(x) = \frac{3x+1}{2(2-x)}$$

$$D_{f^{-1}} = (-\infty, 2) \cup (2, \infty) = E_f$$
$$E_{f^{-1}} = D_f = \left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, \infty\right)$$

.

(b) (i) $f(x) = \ln(x+3)$

$D_f = (-3, \infty)$

To find the range $E_f$, let's find the inverse function first. Because the domain of $f^{-1}$ will be the range of $f$.

(ii)

$$\text{Let } y = \ln(x+3)$$
$$\ln(x+3) = y$$

Using the definition of log function,

$$x + 3 = e^y$$
$$x = e^y - 3$$
$$\implies f^{-1}(x) = e^x - 3$$

$$D_{f^{-1}} = (-\infty, \infty) = E_f$$
$$E_{f^{-1}} = D_f = (-3, \infty)$$