EE361: Control Systems

Homework 5 Solution

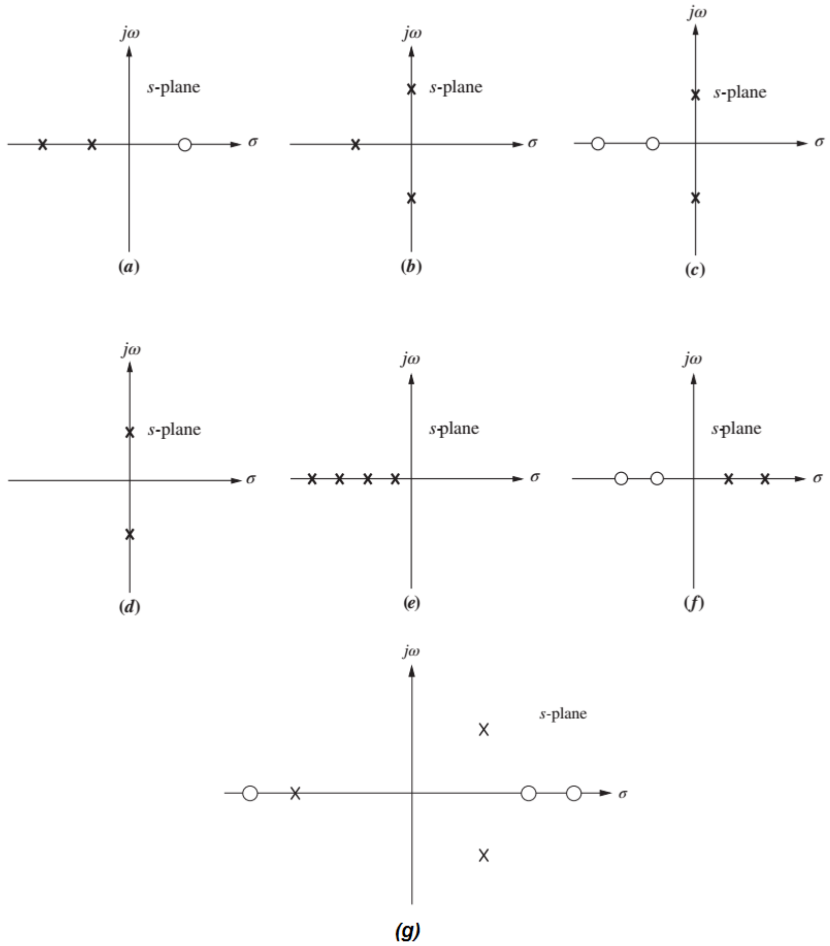
Spring 2019

Problem 1

Problem 1

Each of the following plots shows poles and zeros of an open-loop transfer function $G(s)$ with unity feedback. Sketch the root locus of the closed-loop system for $0 < K < \infty$.

(If we vary K continuously from 0 to ∞, roots of the closed-loop characteristic equation $1 + KG(s) = 0$ change their location and move on a curve called root locus. These roots are actually the location of closed-loop poles of the unity feedback system.)



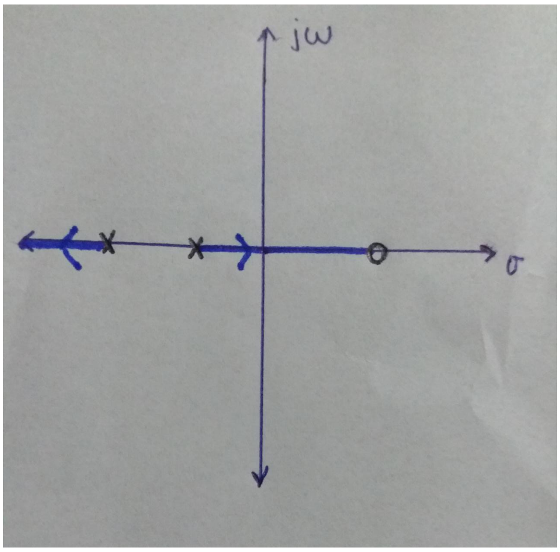


Figure 1.1: (a)

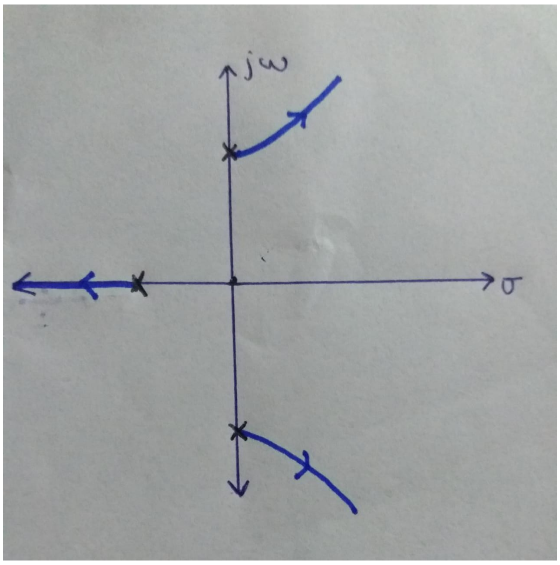


Figure 1.2: (b)

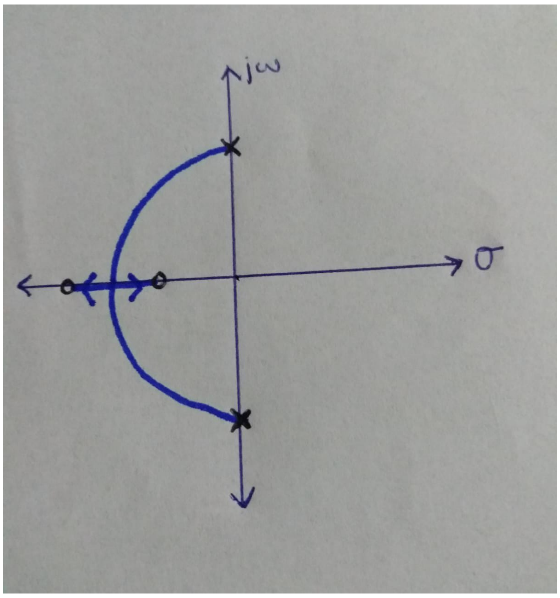


Figure 1.3: (c)

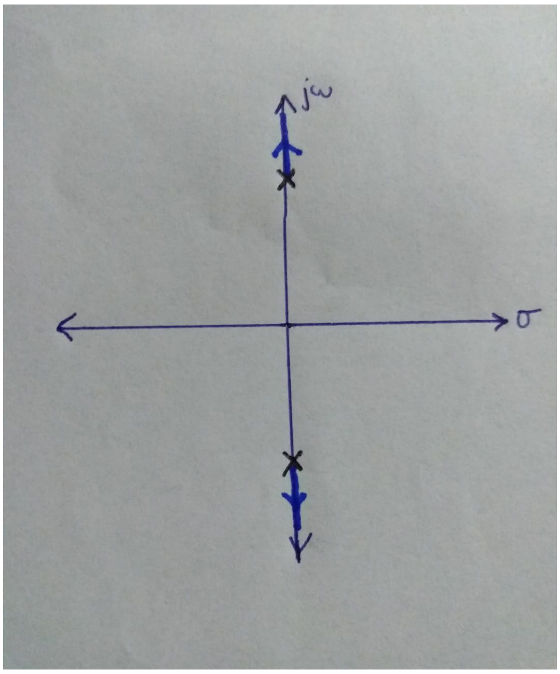


Figure 1.4: (d)

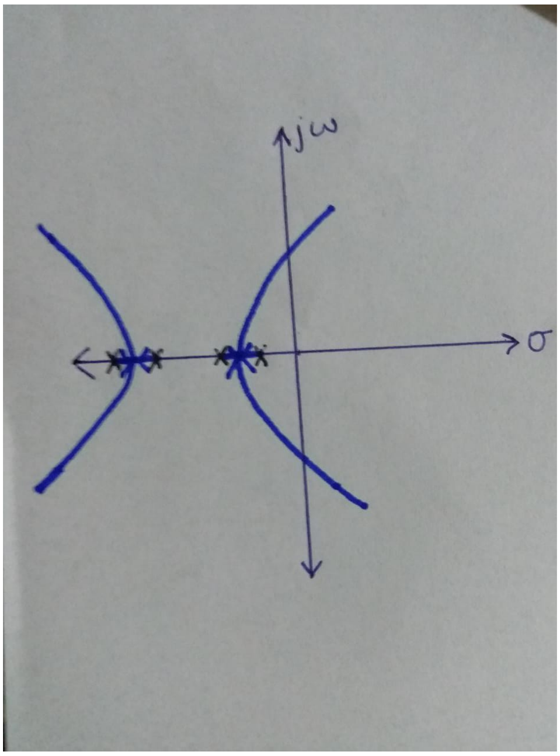


Figure 1.5: (e)

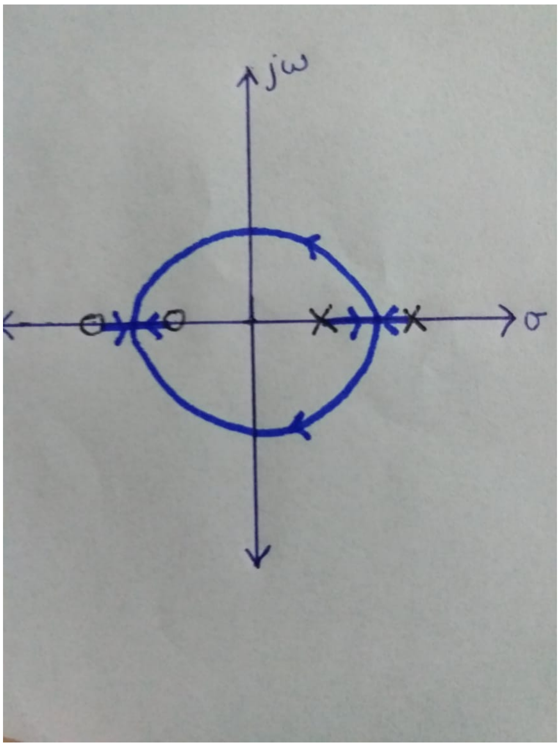


Figure 1.6: (f)

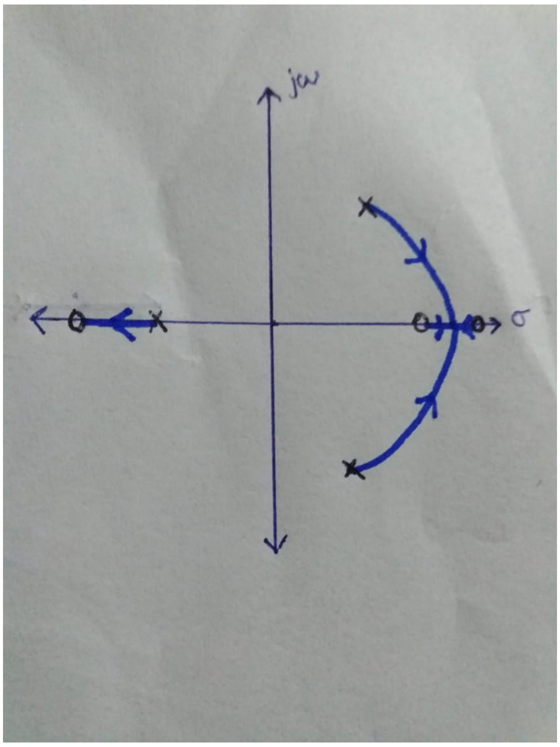
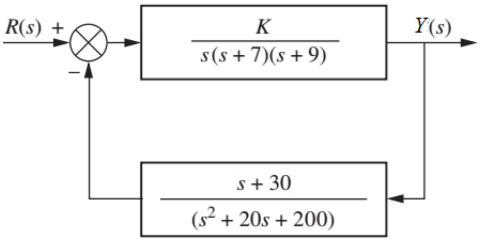


Figure 1.7: (g)

Problem 2

For the following closed-loop system,



(a) Find the range of values of K to yield stability. [Hint: Closed-loop characteristic equation]
The over-all transfer function of the system is given by

$$T(s) = \frac{G(s)}{1 + H(s)G(s)}$$

$$T(s) = \frac{K(s^2 + 20s + 200)}{s(s+7)(s+9)(s^2 + 20s + 200) + K(s+30)}$$

$$T(s) = \frac{K(s^2 + 20s + 200)}{s^5 + 36s^4 + 583s^3 + 4460s^2 + (12600 + K)s + 30K}$$

In order to find the range of K that yield stability we need to create Routh-Hurwitz table:

s^5	1	583	12600+K
s^4	36	4460	30K
s^3	$\frac{4132}{9}$	$\frac{K}{6}+12600$	0
s^2	$\frac{3586580}{1033}$-$\frac{27K}{2066}$	30K	0
s^1	$\frac{9K^2+55200760K-180763632000}{54K-14346320}$	0	0
s^0	30K	0	0

Condition-1 : $\frac{3586580}{1033} - \frac{27K}{2066} > 0$
$\implies K < 265672.59$

Condition-2 : $\frac{9K^2+55200760K-180763632000}{54K-14346320} > 0$
$\implies K < 3272.9119, K > -6136690.639$

Condition-3 : $30K > 0 \implies K > 0$

Combining all the conditions we get

$$0 < K < 3273.9119 \quad (1.1)$$

(b) Is there a value of K for which the system's step response will be undamped? If yes, find that value.
System response gets undamped when poles lie on the imaginary axis. The value of K for which the root locus intersects the imaginary axis is the value at which systems's step response will become undamped. From equation-1.1, for $K = 3272.9119$ the response of the system will become undamped.

(c) Sketch the root locus of the system.[Hint: Convert the closed-loop characteristic equation to the form $1 + KT_{\text{OL}}(s) = 0$, where transfer function $T_{\text{OL}}(s)$ is your equivalent open-loop transfer function with unity feedback.]
The close loop characteristic equation is given by

$$s(s+7)(s+9)(s^2+20s+200) + K(s+30)$$

We need to convert it into $1 + KL(s) = 0$ form.

$$1 + K\frac{s+30}{s(s+7)(s+9)(s^2+20s+200)}$$

Hence

$$G(s) = \frac{s+30}{s(s+7)(s+9)(s^2+20s+200)}$$

$$G(s) = \frac{s+30}{s(s+7)(s+9)(s+10+10j)(s+10-10j)}$$

$$Poles: 0, -7, -9, -10 \pm 10j$$

$$Zeros: -30, \infty, \infty, \infty, \infty$$

The real axis asymptotes are given as

$$\sigma_a = \frac{\sum finite\ poles - \sum finite\ zeros}{n_{finitepoles} - n_{finitezeros}} = -4$$

The angle of the asymptotes are given by

$$\theta_a = \frac{(2l+1)180}{n_{finitepoles} - n_{finitezeros}} = 45, -45, 135, 225$$

The root locus of the system is given by

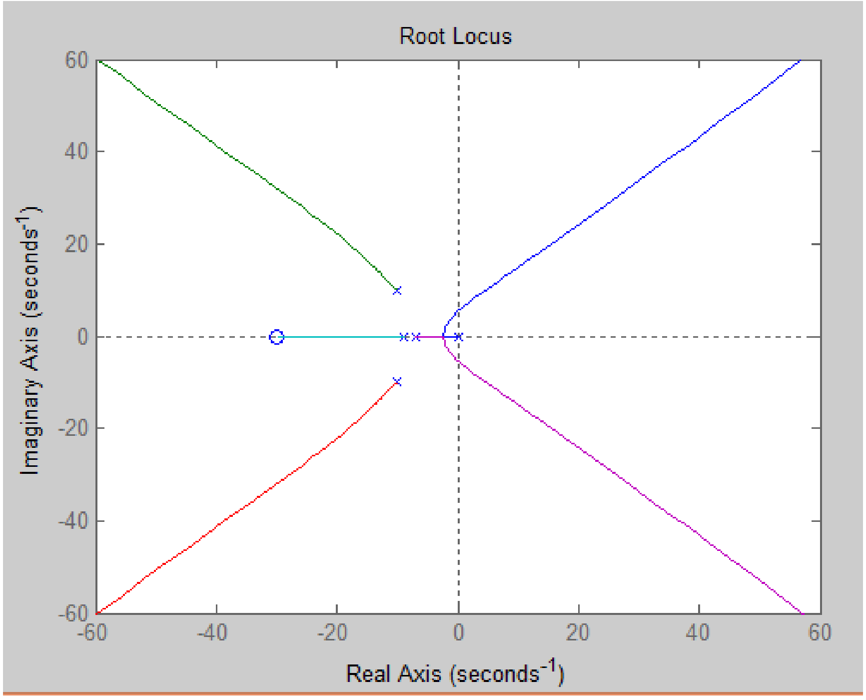


Figure 1.8

(d) Find the value of K that will yield an overshoot of 5% of the step response of the system's dominant poles. (You can solve this part using `rlocus()` plot on MATLAB but you must mathematically verify that at this value, the step response of the system's dominant poles have an overshoot of 5%.)

From root locus graph we can find that for $\zeta = 0.672,\ K = 683$

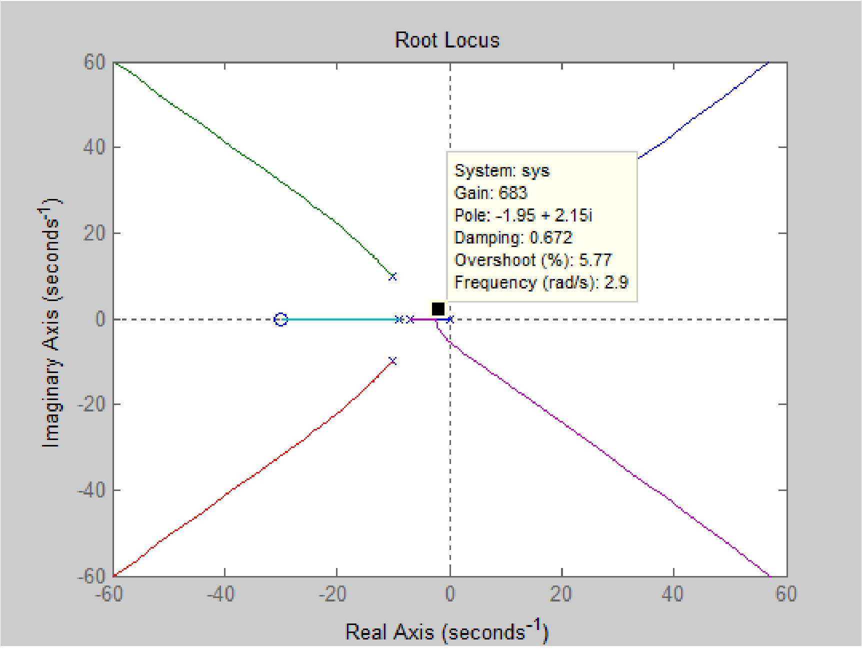


Figure 1.9

Mathematical Justification

The characteristic equation is given by

$$s^5 + 36s^4 + 583s^3 + 4460s^2 + (12600 + K)s + 30K$$

For $K = 683$, the roots of the equation are

$$-10.3291 + 10.3128j, -10.3291 - 10.3128j, -11.4425, -1.95 + 2.145j, -1.95 + 2.145j$$

Hence the dominant poles are **-1.95+2.145j,-1.95-2.145j**

The damping ratio can be evaluated using the following formula

$$\zeta = \frac{\sigma}{\omega_n}$$

$$\zeta = \frac{1.95}{\sqrt{1.95^2 + 2.145^2}} = 0.672$$

(e) Find the value of K that will yield closed-loop poles that give approximately critically damped response. (You can solve this part using `rlocus()` plot on MATLAB but you must mathematically verify that at this value, the system's dominant poles have a critically damped response.)

The response of the system is critically damped whenever we have repeated poles. From the root locus diagram we can see that there is one point where repeated poles can exist

For $K = 416.5$, repeated poles exist at -2.3973

Mathematical Justification

The characteristic equation is given by

$$1 + KH(s)G(s) = 0$$

Find the roots of the equation at $K = 416.5$

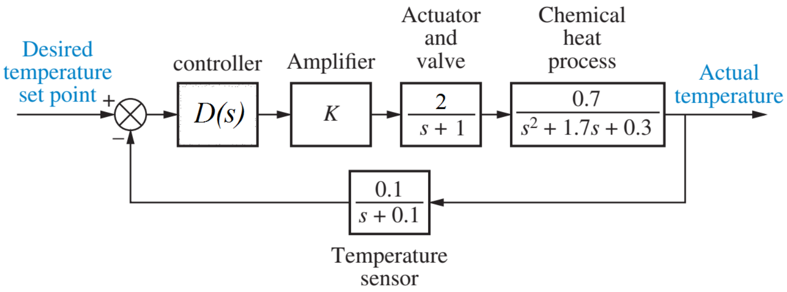
$$s^5 + 36s^4 + 583s^3 + 4460s^2 + 13016.5s + 12486 = 0$$

The roots of the equation are

$$-10.2131 + 10.1964i, -10.2131 - 10.1964i, -10.8755 + 0.0000i, -2.3973 + 0.0000i, -2.3011 + 0.0000i,$$

Problem 3

Consider the temperature control system for a chemical process. You have already analyzed this system in Homework 4.



The system without any compensation ($D(s) = 1$) is operating with a 20% overshoot and a peak time of 14 seconds. There is also a considerable steady-state error.

(a) Estimate the value of K at the uncompensated operating specifications given above.

$$T(s) = \frac{kG(s)}{1 + kG(s)H(s)}$$

$$T(s) = \frac{\dfrac{1.4k}{(s+1)(s^2+1.7s+0.3)}}{1 + \dfrac{0.14k}{(s+1)(s^2+1.7s+0.3)(s+0.1)}}$$

Characteristic equation is

$$1 + k\frac{0.14}{(s+1)(s^2+1.7s+0.3)(s+0.1)} = 0$$

So we need to plot the root locus with respect to the following transfer function

$$\frac{0.14}{(s+1)(s^2+1.7s+0.3)(s+0.1)}$$

Now using the given specifications,

$$OS = 0.2 \implies \zeta = \sqrt{\frac{(\ln 0.2)^2}{\pi^2 + (\ln 0.2)^2}} = 0.45$$

$$T_p = 14 \implies \omega_d = \frac{\pi}{T_p} = 0.224$$

$$\sigma_d = \omega_d \tan(\sin^{-1}\zeta) = 0.1128$$

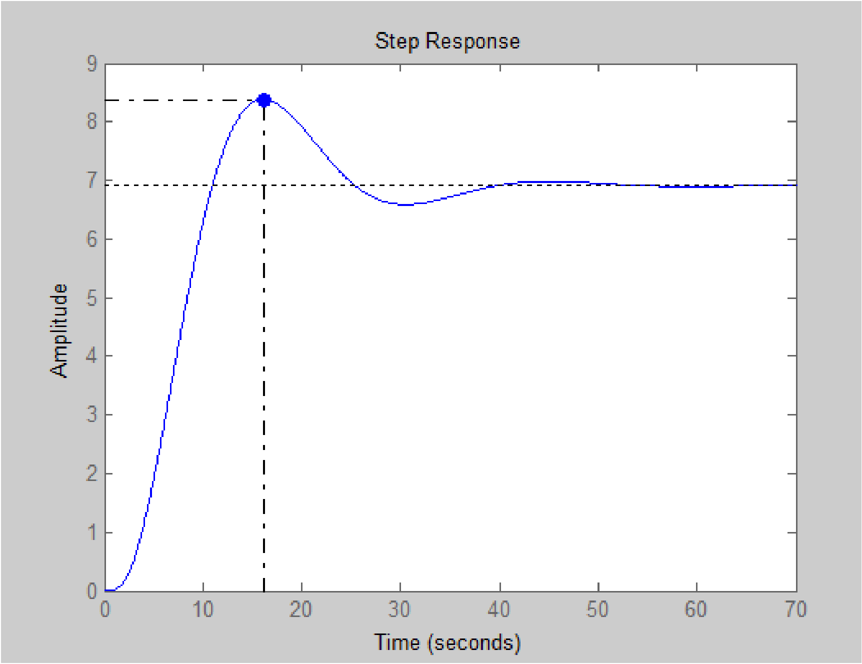
So the two dominant closed-loop poles must lie at $s = -0.1128 \pm 0.224j$ Now to check whether any of these poles lie on the root locus, at this point $\angle G(s)H(s) = 180°$ For value of k,

$$K = \frac{1}{|G(s)H(s)|}$$

$$K = \frac{1}{|G(-0.1128 + 0.224j)H(-0.1128 + 0.224j)|}$$

$$K = \frac{1}{2.0186} = 0.495$$

Plugging $K = 0.495$ in $T(s) = \dfrac{KG(s)}{1 + kG(s)H(s)}$, we get the following step response of $T(s)$, which match our specifications.



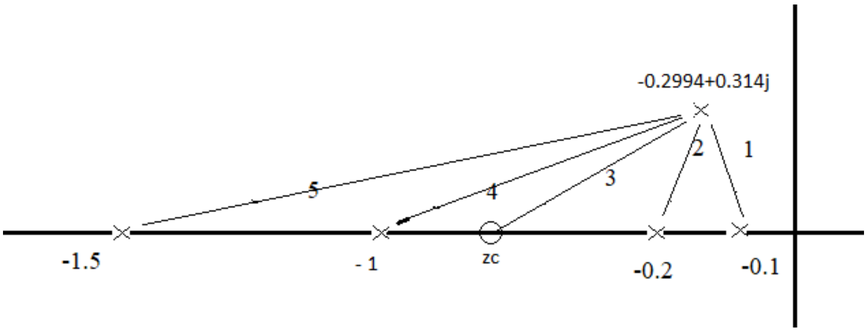
(b) Design a PID controller so that the compensated system will have a peak time approximately 10 s and 5% overshoot to a unit step input. Attach the graphs of all your designed root locii and final step response. (Assume $K = 1$ for this part.)

$$T_p = 10s \implies \omega_d = \frac{\pi}{T_p} = 0.314$$
$$\%OS = 5\% \implies \zeta = 0.691$$
$$\sigma_d = \omega_d \tan(\sin^{-1}\zeta) = 0.2994$$

For pole at $s = -0.2994 \pm 0.314j$, design a PD controller first

$$D(s) = k_d\left(s + \frac{k_p}{k_d}\right)$$

Characteristic equation $1 + D(s)G(s)H(s)$ has poles at $s = -0.1, -0.2, -1, -1.5$ and zero at $s = -\dfrac{k_p}{k_d}$. The desired pole is at $-0.2994 + 0.314j$



We know that on the root locus,

$$sum of angles of zeros - sum of angles of poles = 180\,\text{deg}$$

$$\implies \theta_3 = 180 + \theta_1 + \theta_2 + \theta_4 + \theta_5$$

$$\theta_1 = 180 + \tan^{-1}\left(\frac{0.314}{-0.2994 + 0.1}\right) = 122.41\,\text{deg}$$

$$\theta_2 = 180 + \tan^{-1}\left(\frac{0.314}{-0.2994 + 0.2}\right) = 107.56\,\text{deg}$$

$$\theta_4 = \tan^{-1}\left(\frac{0.314}{-0.2994 + 1}\right) = 24.14\,\text{deg}$$

$$\theta_5 = \tan^{-1}\left(\frac{0.314}{-0.2994 + 1.5}\right) = 14.65\,\text{deg}$$

$$\theta_3 = 180 + 107.56 + 14.65 + 24.14 + 122.41 = 448.73\,\text{deg}$$

$$\theta_3 = 448.73 - 360 = 88.73\,\text{deg}$$

Let the unknown location of zero be z_c.

$$-0.2994 + z_c = \frac{0.314}{\tan 88.73} \implies z_c = 0.3036$$

so,

$$\frac{k_p}{k_d} = 0.3036$$

Now let's design a PI controller:

$$D(s) = \frac{k_p\left(s + \frac{k_i}{k_p}\right)}{s}$$

Its pole is at $s = 0$, so zero must be chosen near the origin so that our PD design is not affected much. Let zero is at $s = -z_2$ and $z_2 = 0.065$.

Now combining the two,

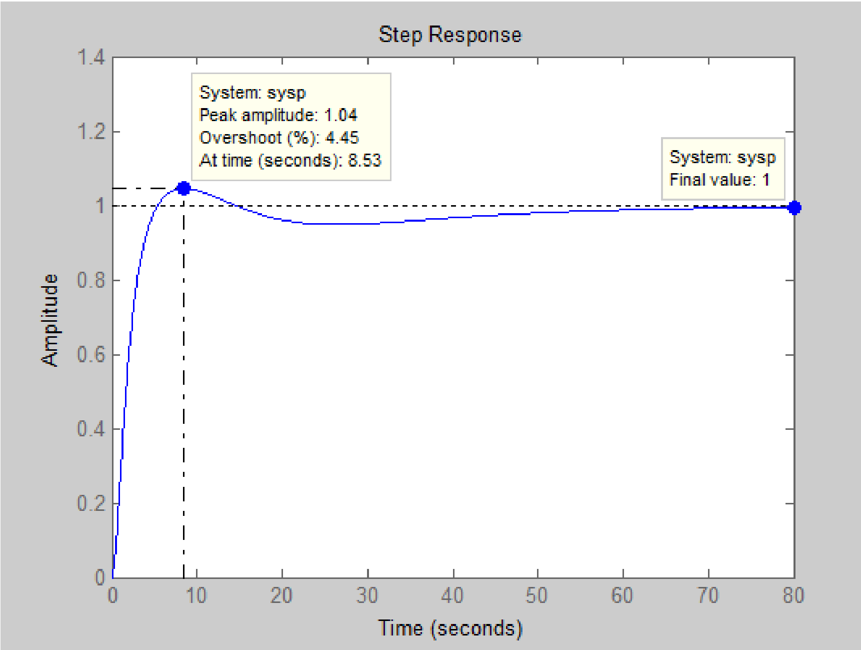
$$D(s) = \frac{k(s + z_c)(s + z_2)}{s} = \frac{k(s + 0.3036)(s + 0.065)}{s} = \frac{k(s^2 + 0.3686s + 0.01973)}{s}$$

$$k = \frac{1}{\left|\frac{(s^2+0.3686s+0.01973)}{s}G(s)H(s)\right|}\Bigg|_{s=-0.2994+0.314j} = 2.65$$

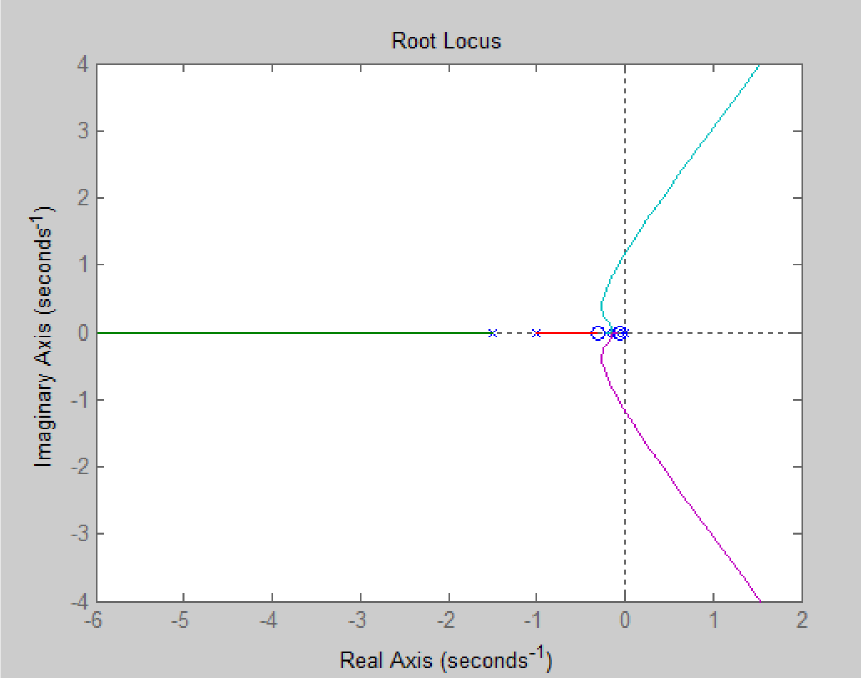
$$D(s) = 2.65(s^2 + 0.3686s + 0.01973)$$

But for this controller, peak time and overshoot are not according to our requirement.

Hence we'll need to adjust the value of k using trial and error. By adjusting value of k to 0.81 we can achieve the given specfications of overshoot and peak time.



Root locus of the system after implementing PID controller is:



(c) Now design a lag-lead compensator to meet the specifications in (b) and reduce the steady-state error to 10% of its original value. Attach the graphs of all your designed root locii and final step response. To calculate the steady state error of the uncompensated system convert the system into corresponding unity feedback system.Open loop transfer function for the corresponding unity feedback system is

$$T_{OL}(s) = \frac{G(s)}{1 + (1 - H(s)G(s)}$$

$$T_{OL}(s) = \frac{0.688s^4 + 1.926s^3 + 1.562s^2 + 0.344s + 0.02064}{s^7 + 5.5s^6 + 11.83s^5 + 11.84s^4 + 4.902s^3 + 0.386s^2 + 0.0036s + 0.009}$$

Steady state error for type-0 system is

$$e_{ss} = \frac{1}{1+K_p}$$

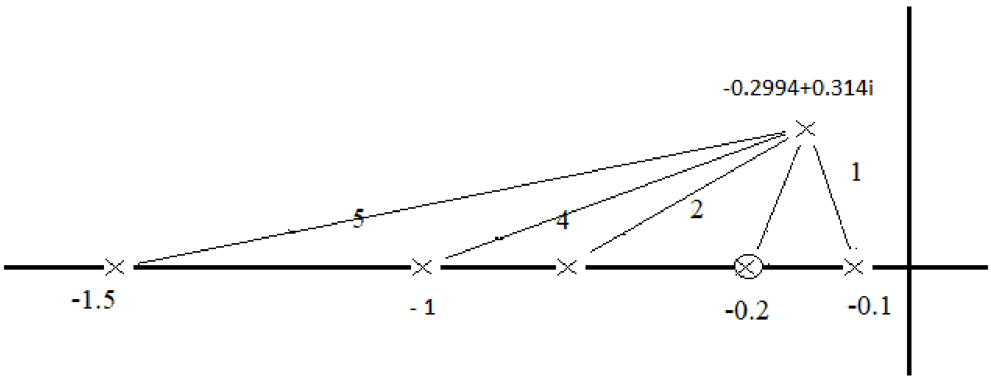
$$K_p = \lim_{s\to 0} T_{OL}(s) = 2.29$$

$$e_{ss} = 0.3036$$

Required steady state error is $e_{ss} = 0.03036$.
Now design lead compensator.

$$D_{lead}(s) = \frac{k(s+z_1)}{(s+p_1)}$$

Place z_1 at $s = -0.2$ to cancel the dominant pole at -0.2.



$$\theta_1 = 180 + \tan^{-1}\left(\frac{0.314}{-0.2994+0.1}\right) = 122.41\,\text{deg}$$

$$\theta_4 = \tan^{-1}\left(\frac{0.314}{-0.2994+1}\right) = 24.14\,\text{deg}$$

$$\theta_5 = \tan^{-1}\left(\frac{0.314}{-0.2994+1.5}\right) = 14.65\,\text{deg}$$

$$\theta_2 = 180 + 14.65 + 24.14 + 122.41 = 341.206\,\text{deg}$$

$$\theta_3 = 360 - 341.206 = 18.794\,\text{deg}$$

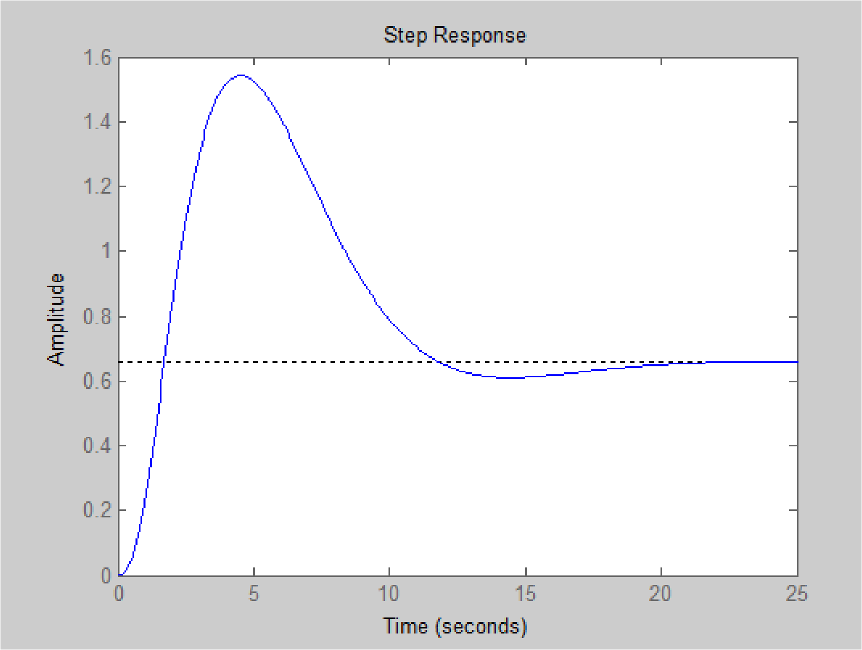
$$-0.2994 + p_1 = \frac{0.314}{\tan 18.794} \implies p_1 = 1.2221$$

$$D_{lead}(s) = k\frac{s+0.2}{s+1.2221}$$

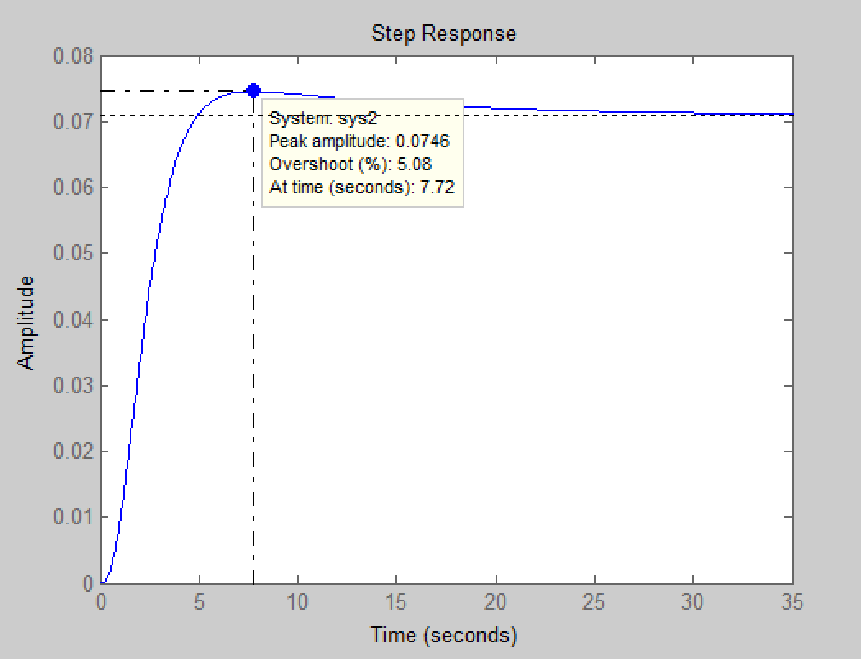
Now find k
$k = \dfrac{1}{D(s)G(s)H(s)}$ at $s = -0.2994 + 0.314j$

$$k = 2.5$$

we have to adjust k to reach required peak time.



By hit and trail, $k = 0.1$



Now design Lag compensator to reduce steady state error.

$$D_{lag}(s) = \frac{s + z_2}{s + p_2}$$

Raito between the required strady state error and the error after implementing lead compensator is 30.36. Place the pole $p_2 = 0.0036$ and zero $z_2 = 0.1$ to keep the raito constant (In this case it is approximately 27).So the lag compensator is:

$$D_{lag}(s) = \frac{s + 0.1}{s + 0.0036}$$

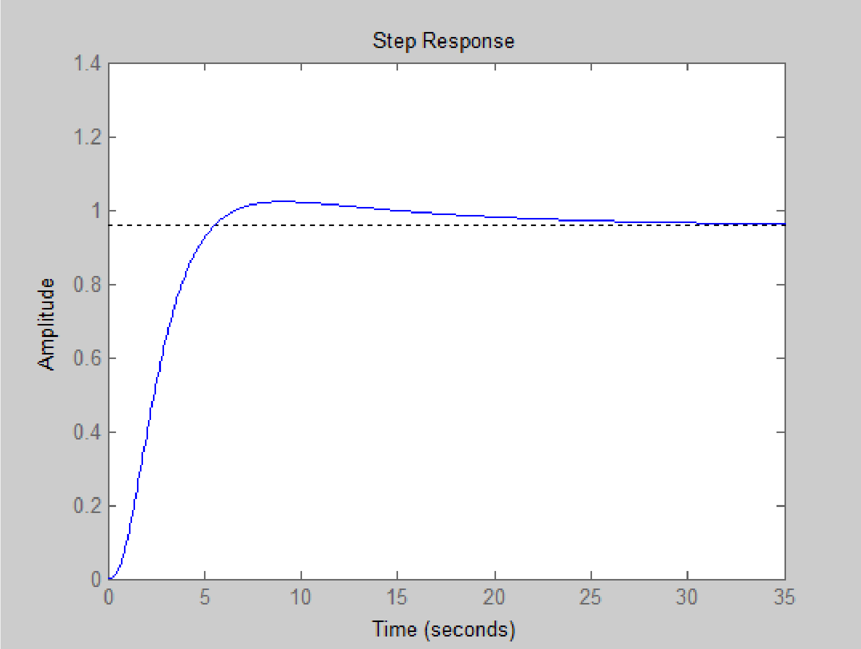
Combining Lag and Lead compensators:

$$D(s) = \frac{(s+0.1)(s+0.2)}{(s+0.0036)(s+1.2221)}$$

Now find k
$k = \frac{1}{D(s)G(s)H(s)}$ at $s = -0.2994 + 0.314j$

$$k = 2.89$$

we have to adjust k to reach required peak time.By hit and trail, $k = 1.09$.Now for compensated sytem is $e_{ss} = 0.04$,$T_p = 8.5sec$ and $\%OS = 5.5\%$



Final root locus of the system after implementing Lag-lead control is