**EE361: Control Systems**

# Homework-3 Solution

*Due Fri Mar 22, 2 PM*

*Spring 2019*

## Problem 1

The response of the deflection of a fluid-filled catheter to changes in pressure can be modeled using a second-order model. Knowledge of the parameters of the model is important because in cardiovascular applications the undamped natural frequency should be close to five times the heart rate. However, due to sterility and other considerations, measurement of the parameters is difficult. A method to obtain transfer functions using measurements of the amplitudes of two consecutive peaks of the response and their timing has been developed. Assume that Figure 2 is obtained from catheter measurements.

Using the information shown and assuming a second-order model excited by a unit step input, find the corresponding transfer function.

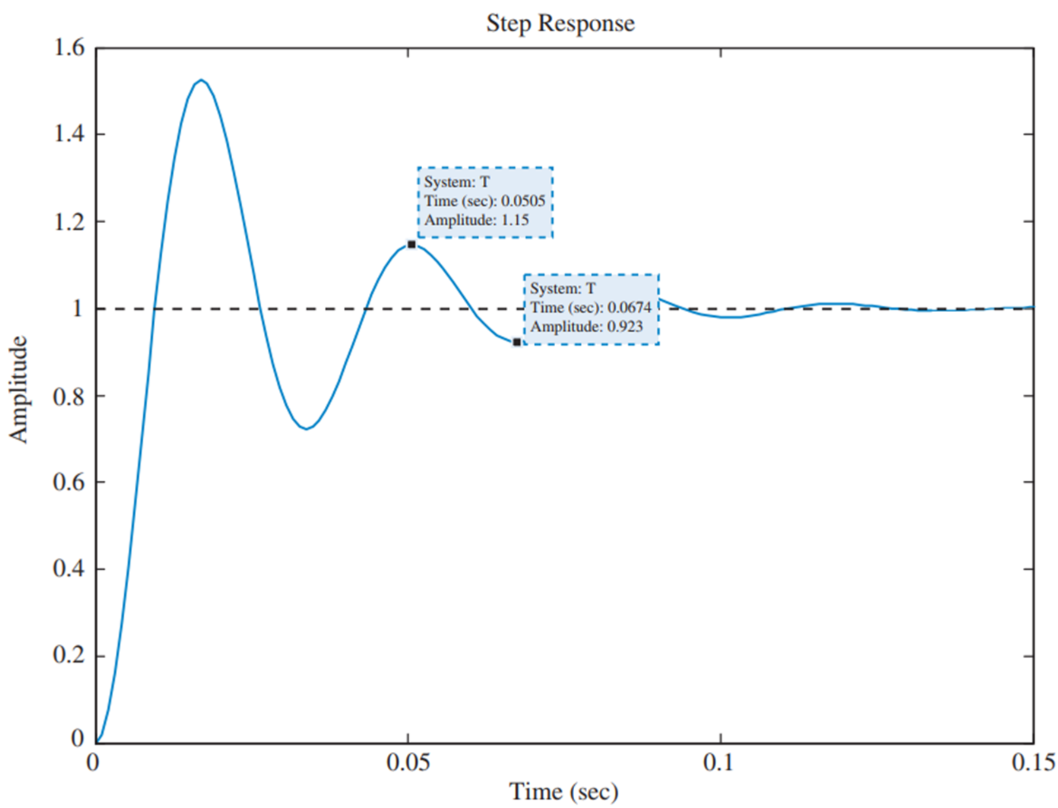


Figure 1

**Solution**

We know that oscillation period is given by

$$\frac{2\pi}{T} = \omega_n \sqrt{1 - \zeta^2} \tag{1.1}$$

From the graph given, the time period of half wavelength is given by

$$\frac{T}{2} = 0.0674 - 0.0505 = 0.0169$$

$$T = 2 \times 0.0169 = 0.0338$$

Using equation 1.1

$$\frac{2\pi}{0.0338} = \omega_n \sqrt{1-\zeta^2}$$
$$\omega_n \sqrt{1-\zeta^2} = 185.36 \tag{1.2}$$

Referring to equation 4.28 in the book

$$c(t) = 1 - \frac{e^{-\zeta\omega t}}{\sqrt{1-\zeta^2}} \cos\left(\omega_n \sqrt{1-\zeta^2} t - \phi\right)$$

The peaks of the response occur whenever cos term of the step response is $\pm 1$ From figure it is clear that

$$1 + \frac{e^{-\zeta\omega(0.0505)}}{\sqrt{1-\zeta^2}} = 1.15$$
$$\frac{e^{-\zeta\omega(0.0505)}}{\sqrt{1-\zeta^2}} = 0.15 \tag{1.3}$$

From the second point other graph given in the figure

$$1 - \frac{e^{-\zeta\omega(0.0674)}}{\sqrt{1-\zeta^2}} = 0.923$$

$$\frac{e^{-\zeta\omega(0.0674)}}{\sqrt{1-\zeta^2}} 0.077 \tag{1.4}$$

Now, dividing equation 1.3 and 1.4

$$\frac{e^{-\zeta\omega(0.0505)}}{e^{-\zeta\omega(0.0674)}} = \frac{0.15}{0.707}$$

$$e^{\zeta\omega(0.0169)} = 1.9480$$

$$\ln\left(e^{-\zeta\omega(0.0169)}\right) = \ln(1.9480)$$
$$\zeta\omega_n = 39.45$$
$$\omega_n = \frac{39.45}{\zeta}$$

Using equation 1.2

$$\omega_n \sqrt{1-\zeta^2} = 185.36$$

$$\frac{39.45}{\zeta} \sqrt{1-\zeta^2} = 185.36$$
$$\zeta = 0.2178$$
$$\omega_n = \frac{39.45}{\zeta}$$
$$\omega_n = \frac{39.45}{0.2178} = 181.11$$

Second order transfer function in terms of natural frequency and damping ratio is given by

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$G(s) = \frac{181.11^2}{s^2 + (2 \times 0.2178 \times 181.11)s + 181.11^2}$$

$$G(s) = \frac{32800.8}{s^2 + 78.89s + 32800.8}$$

**Solution-2**

An approximate solution can also be found by using the following strategy.
From the graph, first find the percentage overshoot.

$$\%OS = \frac{c_{max} - c_{final}}{c_{final}} \times 100$$

Evaluate the damping ratio using equation 4.39 in the book

$$\zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}}$$

Evaluate the peak time ($T_p$) from the graph. After that natural frequency $\omega_n$ can be evaluated using the following formula

$$T_p = \frac{\pi}{\omega_n \times \sqrt{1-\zeta^2}}$$

# Problem 2

A human responds to a visual cue with a physical response,as shown in Figure 1.The transfer function that relates the output physical response $P(s)$, to the input visual command, $V(s)$ is

$$G(s) = \frac{s+1}{s^2 + 7s + 10}$$

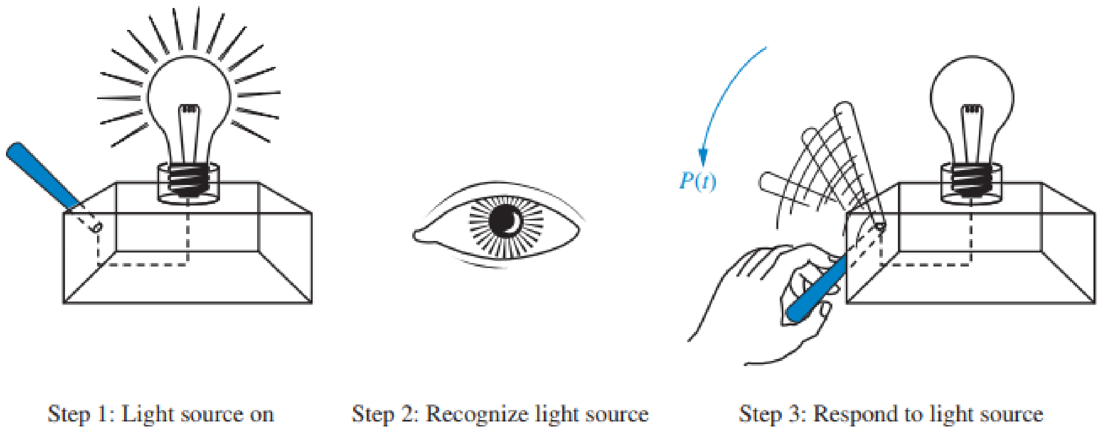


Figure 2

(i) Plot poles and zeros of the system in the s-plane..

$$G(s) = \frac{s+1}{s^2 + 7s + 10}$$

$$G(s) = \frac{s+1}{s^2+2s+5s+10}$$
$$G(s) = \frac{s+1}{(s+5)(s+2)}$$

Hence poles: $-5, -2$, zeros: $-1$

(ii) Evaluate the output response for a unit step input using the Laplace transform.

As we know that

$$G(s) = \frac{s+1}{s^2+7s+10}$$

The step response of the system is given by

$$Y(s) = \frac{s+1}{s^2+7s+10}\frac{1}{s}$$

$$Y(s) = \frac{s+1}{s(s+2)(s+5)}$$

Using the partial fractions

$$\frac{s+1}{s(s+2)(s+5)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+5}$$

Taking Inverse Laplace Transform

$$= \frac{\frac{1}{10}}{s} + \frac{\frac{1}{6}}{s+2} + \frac{\frac{-4}{5}}{s+5}$$

$$y(t) = \frac{1}{10} + \frac{1}{6}e^{-2t} - \frac{4}{5}e^{-5t}$$

(iii) Represent the transfer function in state space.

$$G(s) = \frac{s+1}{s^2+7s+10}$$
$$G(s) = \frac{1+s}{10+7s+s^2}$$

We know that

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$

$$\mathbf{y} = C\mathbf{x} + Du$$
$$A = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = 0$$

(iv) What are the eigenvalues of the matrix A?

The characteristic equation is given by

$$det(\lambda I - A) = (\lambda+2)(\lambda+5) = 0$$
$$\implies \lambda = -2, -5$$

(v) Use MATLAB to simulate the system and attach a plot of the step response.

The step response of the system is given by

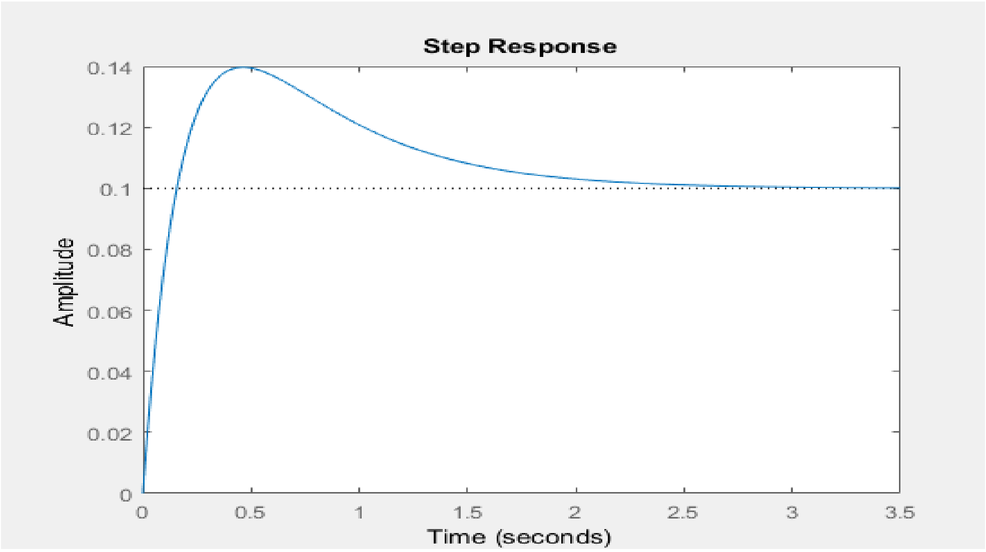


Figure 3

(vi) Repeat parts (a) to (e) for the following transfer function and compare the graphs.

$$G(s) = \frac{s-1}{s^2+7s+10}$$

(vii) Plot poles and zeros of the system in the s-plane.
Poles: $-5, -2$, zeros: $+1$

(viii) Evaluate the output response for a unit step input using the Laplace transform. As we know that

$$G(s) = \frac{s-1}{s^2+7s+10}$$

The step response of the system is given by

$$Y(s) = \frac{s-1}{s^2+7s+10}\frac{1}{s}$$

$$Y(s) = \frac{s-1}{s(s+2)(s+5)}$$

Using the partial fractions

$$\frac{s-1}{s(s+2)(s+5)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+5}$$

$$= \frac{-\frac{1}{10}}{s} + \frac{\frac{1}{2}}{s+2} + \frac{-\frac{2}{5}}{s+5}$$

Taking Inverse Laplace Transform

$$y(t) = -\frac{1}{10} + \frac{1}{2}e^{-2t} - \frac{2}{5}e^{-5t}$$

(ix) Represent the transfer function in state space.

$$G(s) = \frac{s-1}{s^2+7s+10}$$

$$G(s) = \frac{1-s}{10+7s+s^2}$$

We know that

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$

$$\mathbf{y} = C\mathbf{x} + Du$$

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 1 \end{bmatrix}, D = 0$$

(x) What are the eigenvalues of the matrix A?
The characteristic equation is given by

$$det(\lambda I - A) = (\lambda + 2)(\lambda + 5) = 0$$

$$\implies \lambda = -2, -5$$

(xi) Use MATLAB to simulate the system and attach a plot of the step response.
The step response of the system is given by

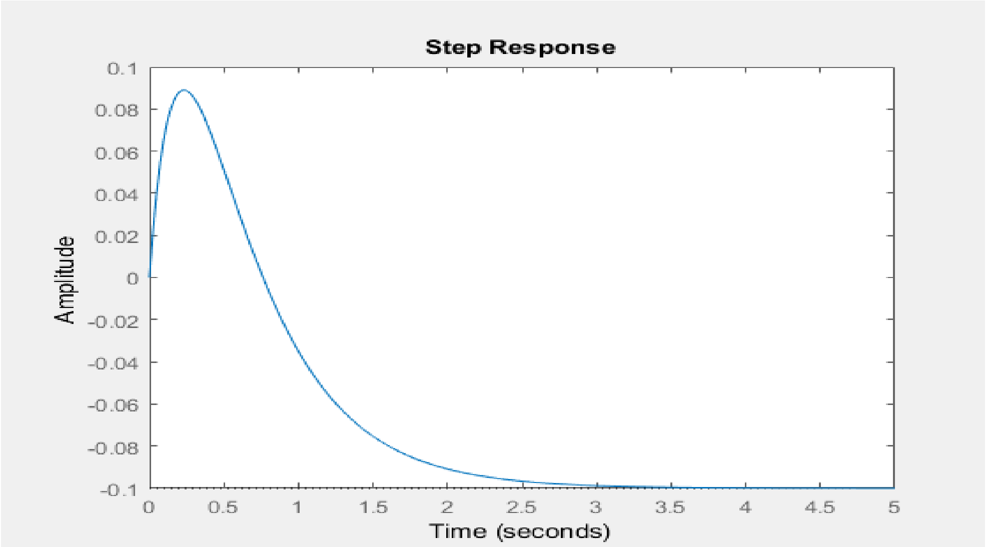


Figure 4

# Problem 3

Consider the following transfer function.

$$G(s) = \frac{s^3 + 4s^2 + 3s + 1}{s^4 + s^3 + 6s^2 + 26s + 20}$$

(i) Using Routh's table, determine how many of its poles do not lie in the left half plane?

Applying Routh-Hurwitz criterion to the denominator of transfer function. The routh table is given by

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 6 | 20 |
| $s^3$ | 1 | 26 | 0 |
| $s^2$ | -20 | 20 | 0 |
| $s^1$ | 27 | 0 | 0 |
| $s^0$ | 20 | 0 | 0 |

It is evident from the first column of the table that, there are two sign changes. Hence two poles will lie in the right half plane and two poles will reside in the left half plane.

(ii) What is the value of $\lim_{\mathcal{T}\to\infty} \int_0^{\mathcal{T}} |g(t)|dt$? Explain
Poles in the right half-plane yield either pure exponentially increasing or exponentially increasing sinusoidal natural responses, hence natural responses approach infinity ($g(t) \to \infty$) as time approaches infinity ($t \to \infty$ ). As two of the poles of this system lies in the right half plane, this implies that the natural response of the system approach infinity ($g(t) \to \infty$) as time approaches infinity ($t \to \infty$ ). This would mean that the impulse response of the system is not absolutely integrable. The impulse response would be $\lim_{\mathcal{T}\to\infty} \int_0^{\mathcal{T}} |g(t)|dt = \infty$

(iii) Is this system internally stable? Explain.

As the impulse response of the system is not absolutely integrable, the system is internally unstable.

(iv) Is this system BIBO stable? Explain.
As the system is not internally stable, it would not be BIBO stable.