EE361: Control Systems

Homework 2 Solution

Due Fri Mar 8, 2 PM

Spring 2019

Problem 1

Consider the mass-spring system shown in the figure below. Deja vu? Yes, this is the same system that you dealt with in Homework 1.

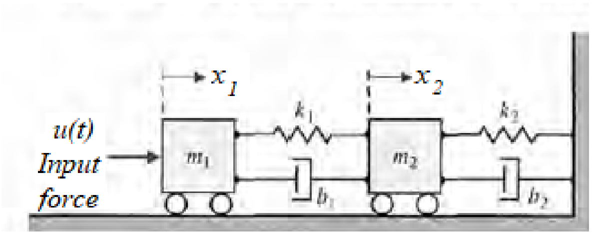

Figure 1: Double mass-spring system

Two carts of masses m_1 and m_2 with negligible rolling friction are connected to two springs (spring constant k_1 and k_2) and dampers (damping constant b_1 and b_2) as shown in the figure. An input force $u(t)$ is applied on m_1 and the output is the position of m_2. The displacements of m_1 and m_2 from their equilibrium positions are x_1 and x_2 respectively.

(a) Write down the differential equation for the displacement x_1 of mass m_1.

For writing the differential equation for the displacement x_1 of mass m_1 apply the super position principle, m_1 moving m_2 stationary

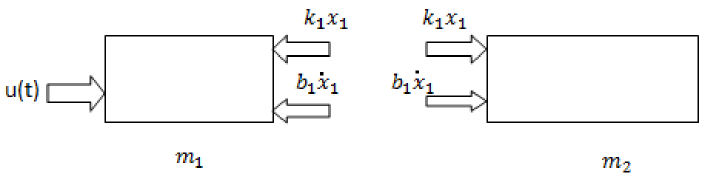

Figure 2: Free Body Diagram of Spring-Mass-Damper System m_1 moving m_2 stationary

Using free body diagram, equation of motion form_1 according to Newton's second law is

$$F_{net} = m_1 a = u(t) - (x_1 - x_2)k_1 - (\dot{x_1} - \dot{x_2})b_1$$

$$m_1\ddot{x_1} = u(t) - (\dot{x_1} - \dot{x_2})b_1 - (x_1 - x_2)k_1 \tag{1.1}$$

(b) Write down the differential equation for displacement x_2 of mass m_2.

For writing the differential equation for the displacement x_2 of mass m_2 apply the super position principle, m_2 moving m_1 stationary

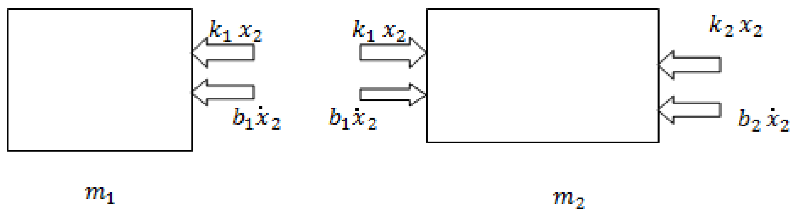

Figure 3: Free Body Diagram of Spring-Mass-Damper System m_2 moving m_1 stationary

Using free body diagram, equation of motion for m_2 according to Newton's second law is

$$F_{net} = m_2 a = -k_2 x_2 - b_2 \dot{x_2} + (x_1 - x_2)k_1 + (\dot{x_1} - \dot{x_2})b_1$$

$$m_2 \ddot{x_2} - (\dot{x_1} - \dot{x_2})b_1 - (x_1 - x_2)k_1 + k_2 x_2 + b_2 \dot{x_2} = 0 \tag{1.2}$$

(c) Given that the force u is the input and displacement x_2 is the output. Using the state vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{bmatrix}$ where p_1 and p_2 are momentums of mass 1 and 2 respectively, write down the state-space representation of the system. Clearly specify the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$.

$p_1 = m_1 \dot{x_1}$ and $p_2 = m_2 \dot{x_2}$ .State space equation is:

$$\dot{x} = \mathbf{A}x + \mathbf{B}u$$

Rearranging the Eq.1.1 and Eq.1.2,

$$m_1 \ddot{x_1} = -k_1 x_1 + k_1 x_2 - b_1 \dot{x_1} + b_1 \dot{x_2} + u(t)$$

$$m_2 \ddot{x_2} = k_1 x_1 - (k_1 + k_2)x_2 + b_1 \dot{x_1} - (b_1 + b_2)\dot{x_2}$$

In terms of momentum

$$\dot{p_1} = -k_1 x_1 + k_1 x_2 - \frac{b_1}{m_1} p_1 + \frac{b_1}{m_2} p_2 + u(t)$$

$$\dot{p_2} = k_1 x_1 - (k_1 + k_2)x_2 + \frac{b_1}{m_1} p_1 - \frac{(b_1 + b_2)}{m_2} p_2$$

so the state space model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{m_1} & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} \\ -k_1 & k_1 & -\frac{b_1}{m_1} & \frac{b_1}{m_2} \\ k_1 & -(k_1 + k_2) & \frac{b_1}{m_1} & \frac{-(b_1 + b_2)}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}0 & 0 & \frac{1}{m_1} & 0\\ 0 & 0 & 0 & \frac{1}{m_2}\\ -k_1 & k_1 & -\frac{b_1}{m_1} & \frac{b_1}{m_2}\\ k_1 & -(k_1+k_2) & \frac{b_1}{m_1} & \frac{-(b_1+b_2)}{m_2}\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix},\ \mathbf{D}=0$$

(d) Why is the state vector in (c) a valid choice? Explain in terms of energies.

Total energy of the system is

$$T_e=\frac{1}{2}m_1\dot{x_1}^2+\frac{1}{2}m_2\dot{x_2}^2+\frac{1}{2}k_1(x_1-x_2)^2+\frac{1}{2}k_2x_2^2$$

$$T_e=\frac{1}{2}\frac{p_1^2}{m_1}+\frac{1}{2}\frac{p_2^2}{m_2}+\frac{1}{2}k_1(x_1-x_2)^2+\frac{1}{2}k_2x_2^2$$

Where

$$K.E=\frac{1}{2}\frac{p_1^2}{m_1}+\frac{1}{2}\frac{p_2^2}{m_2}$$

$$P.E=\frac{1}{2}k_1(x_1-x_2)^2+\frac{1}{2}k_2x_2^2$$

These energies varies directly with x_1, x_2, p_1, p_2.So x_1, x_2, p_1, p_2 are valid state variables for this system.

(e) Using the transfer function $\frac{X_2(s)}{U(s)}$ that you found in Homework 1, find a different state-space representation of the system. Can you figure out the state vector in this case?

Transfer function $\frac{X_2(s)}{U(s)}$ that you found in Homework 1 is

$$\frac{X_2(s)}{U(s)}=\frac{(k_1+b_1s)}{(m_1m_2)s^4+(m_2b_1+m_1b_1+m_1b_2)s^3+(m_2k_1+m_1k_2+m_1k_1+b_2b_1)s^2+(b_2k_1+b_1k_2)s+k_1k_2}$$

$$\frac{X_2(s)}{U(s)}=\frac{\frac{k_1}{m_1m_2}+\frac{b_1}{m_1m_2}s}{s^4+\left(\frac{b_1}{m_1}+\frac{b_1}{m_2}+\frac{b_2}{m_2}\right)s^3+\left(\frac{k_1}{m_2}+\frac{k_2}{m_1}+\frac{k_1}{m_1}+\frac{b_2b_1}{m_1m_2}\right)s^2+\left(\frac{b_2k_1}{m_1m_2}+\frac{b_1k_2}{m_1m_2}\right)s+\frac{k_1k_2}{m_1m_2}}$$

General form:

$$G(s)=\frac{\beta_ms^m+\beta_{m-1}s^{m-1}+........+\beta_1s+\beta_0}{s^n+\alpha_{n-1}s^{n-1}+\alpha_{n-2}s^{n-2}+........+\alpha_1s+\alpha_0}$$

$$G(s)=\frac{\beta_1s+\beta_0}{s^4+\alpha_3s^3+\alpha_2s^2+\alpha_1s+\alpha_0}$$

Where $\alpha_0=\frac{k_1k_2}{m_1m_2}$, $\alpha_1=\frac{b_2k_1}{m_1m_2}+\frac{b_1k_2}{m_1m_2}$, $\alpha_2=\frac{k_1}{m_2}+\frac{k_2}{m_1}+\frac{k_1}{m_1}+\frac{b_2b_1}{m_1m_2}$, $\alpha_3=\frac{b_1}{m_1}+\frac{b_1}{m_2}+\frac{b_2}{m_2}$
$\beta_0=\frac{k_1}{m_1m_2}$, $\beta_1=\frac{b_1}{m_1m_2}$

So the state space model is:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1k_2}{m_1m_2} & -\left(\dfrac{b_2k_1}{m_1m_2}+\dfrac{b_1k_2}{m_1m_2}\right) & -\left(\dfrac{k_1}{m_2}+\dfrac{k_2}{m_1}+\dfrac{k_1}{m_1}+\dfrac{b_2b_1}{m_1m_2}\right) & -\left(\dfrac{b_1}{m_1}+\dfrac{b_1}{m_2}+\dfrac{b_2}{m_2}\right)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix} u$$

$$y = \begin{bmatrix}\dfrac{k_1}{m_1m_2} & \dfrac{b_1}{m_1m_2} & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

We can't compute state vector directly from this state state representation.

(f) Compare your answers to (c) and (d) and explain the differences, if any.

Corresponding to one transfer function there are many state space representations.As you can see from your answer to (c) and (d) that these are two different representations, with different $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ matrices.

Problem 2

A space module of mass m descending on Mars is shown in the following figure. The mechanics of its descent can be modeled in the same way we modeled the rocket's equation in class, considering the rate of change of rocket's momentum.

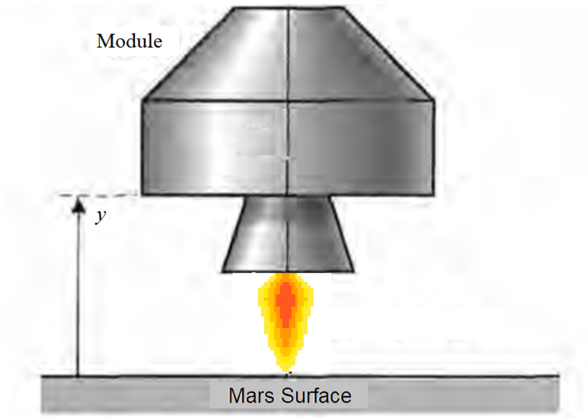

Figure 4: A Space Module Preparing to Land on Mars

The input to this system is the rate at which mass of the fuel burns and is expelled, and the output is the vertical position y of the rocket. Assume that g is the gravity constant on Mars and drag force is proportional to the speed of the module.

(a) What is the order of the system? Explain.

Total energy of the system is

$$T_e = \frac{1}{2}m\dot{y}^2 + mgy$$

There are two energy storing elemets in the system so this is a 2^{nd} order system.

(b) Derive a differential equation for the motion of the space module, in terms of $\dot{y}$, $\ddot{y}$ and $\dot{m}$.

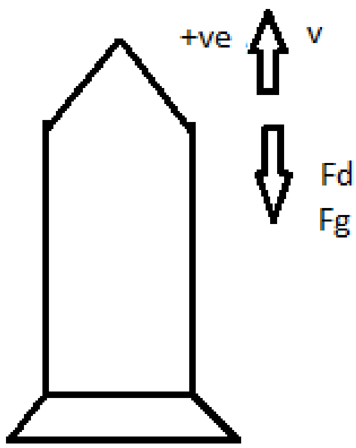

Figure 5: Free body diagram of Space module

Output: vertical height y
Input: $\dot{m}$
Newton's 2^{nd} law in terms of law of conservation of momentum

$$F_{net} = \frac{dp}{dt}$$

$$-F_D - F_g = \frac{d}{dt}mv$$

$$-bv - mg = v\frac{dm}{dt} + m\frac{dv}{dt}$$

$$-b\frac{dy}{dt} - mg = \frac{dy}{dt}\frac{dm}{dt} + m\frac{d^2y}{dt^2}$$

$$-b\dot{y} - mg = \dot{y}\dot{m} + m\ddot{y}$$

(c) Specify a set of appropriate state variables.

Total energy of the system is

$$T_e = \frac{1}{2}m\dot{y}^2 + mgy$$

Where energy varies directly with the rate of change of mass, change in velocity and height of the space module.So appropriate state variables are m, y and $\dot{y}$

(d) Find a state-space model for this system. Is this a linear model? If not, identify all the nonlinearities in the equations.

[Note: In case your system is nonlinear, you do not need to convert the system in the form of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ matrices.]

$$\ddot{y} = -\frac{b}{m}\dot{y} - g - \frac{\dot{y}\dot{m}}{m}$$

The system is non-linear because there exist products of input and state variables.We can't compute $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ matrices directly.