EE361: Control Systems

Homework 1 Solution

Due Thu Feb 21, 2 PM *Spring 2019*

Problem 1

Consider an RC circuit shown in figure below.

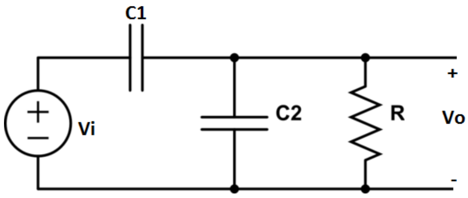

Figure 1: An RC Circuit

(a) Find the transfer function $\frac{V_o(s)}{V_i(s)}$ of the electric circuit using the impedance model.

[Hint: Perform nodal analysis with admittance to make your life easier in this case.]

Convert the circuit into corresponding admittance circuit

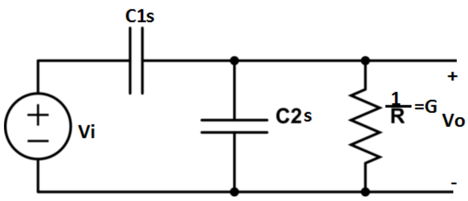

Figure 2: Admittance Circuit

As G is parallel with C_2s

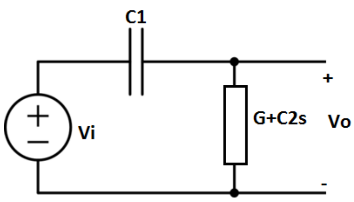

Figure 3: Admittance Circuit

Writing the equation for output node

$$V_o(s)(G + C_2s + C_1s) - C_1sV_{in}(s) = 0 \tag{1.1}$$

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{C_1s}{G + C_2s + C_1s} \tag{1.2}$$

By solving and rearranging the above equation,we get

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{RC_1 s}{(C_1 + C_2)Rs + 1} \tag{1.3}$$

(b) Using your answer to (a), find the output voltage $v_o(t)$ if $R = 100$ kΩ, $C_1 = 500$ nF and $C_2 = 500$ nF for each of the following cases, where the input voltage v_i, measured in volts, is applied at $t = 0$.

(i) $v_i(t) = \delta(t)$

By putting the values in Eq. (1.3),

$$V_{out}(s) = \frac{s}{2s + 20} V_{in}(s) \tag{1.4}$$

We know that Laplace of $\delta(t)$ is 1 so by applying $V_{in}(s)$=1 in Eq. (1.4),

$$V_{out}(s) = \frac{s}{2(s + 10)} \tag{1.5}$$

After converting to partial fraction,

$$V_o(s) = \frac{1}{2} - \frac{5}{s + 10}$$

Taking the inverse Laplace transform,

$$v_o(t) = \frac{1}{2} - 5e^{-10t} \tag{1.6}$$

(ii) $v_i(t) = 1$

Laplace of a constant is $\frac{1}{s}$ so put $V_{in}(s)$=$\frac{1}{s}$ in Eq. (1.3)

$$V_o(s) = \frac{s}{s(2s + 20)}$$

$$V_o(s) = \frac{1}{2(s + 10)}$$

Taking the inverse Laplace transform,

$$v_o(t) = \frac{1}{2} e^{-10t}$$

(iii) $v_i(t) = e^{-10t}$

Laplace transform of e^{-10t} is $\frac{1}{s + 10}$ put this in Eq. (1.3)

$$V_o(s) = \frac{s}{2(s + 10)^2}$$

After converting to partial fraction,

$$V_o(s) = \frac{1}{2(s + 10)} - \frac{5}{(s + 10)^2}$$

Taking the inverse Laplace transform,

$$v_o(t) = \frac{1}{2} e^{-10t} - 5te^{-10t}$$

(iv) $v_i(t) = \sin 2t$

Laplace transform of $\sin 2t$ is $\frac{2}{s^2+4}$ put $V_{in}(s) = \frac{2}{s^2+4}$ in Eq. (1.3)

$$V_o(s) = \frac{2s}{2(s+10)(s^2+4)}$$

$$V_o(s) = \frac{s}{(s+10)(s^2+4)}$$

After converting to partial fraction,

$$V_o(s) = \frac{5s+2}{52(s^2+4)} - \frac{5}{52(s+10)}$$

$$V_o(s) = \frac{5s}{52(s^2+4)} + \frac{2}{52(s^2+4)} - \frac{5}{52(s+10)}$$

Inverse Laplace of $\frac{s}{(s^2+4)} = \cos 2t$, $\frac{2}{(s^2+4)} = \sin 2t$ so

$$v_o(t) = \frac{5}{52}\cos 2t + \frac{1}{52}\sin 2t - \frac{5}{52}e^{-10t}$$

(v) $v_i(t) = 15 - 5e^{-10t} + 50\sin 2t$

[Hint for (v): Apply linearity to the 'particular solutions' in (ii), (iii) and (iv).]

Use the superposition property of linear systems.

$$v_{i1}(t) = 15$$

Laplace of a 15 is $\frac{15}{s}$ so put $V_{in}(s)$=$\frac{15}{s}$ in Eq. (1.3)

$$V_{o1}(s) = \frac{15s}{s(2s+20)}$$

$$V_{o1}(s) = \frac{15}{2(s+10)}$$

Taking the inverse Laplace transform,

$$v_{o1}(t) = \frac{15}{2}e^{-10t} \tag{1.7}$$

Now

$$v_{i2}(t) = -5e^{-10t}$$

Laplace transform of $5e^{-10t}$ is $\frac{5}{s+10}$ put this in Eq. (1.3)

$$V_{o2}(s) = \frac{5s}{2(s+10)^2}$$

After converting to partial fraction,

$$V_{o2}(s) = \frac{25}{(s+10)^2} - \frac{5}{2(s+10)}$$

Taking the inverse Laplace transform,

$$v_{o2}(t) = 25te^{-10t} - \frac{5}{2}e^{-10t} \tag{1.8}$$

Now

$$v_{i3}(t) = 50 \sin 2t$$

Laplace transform of $50 \sin 2t$ is $\dfrac{100}{s^2+4}$ put $V_{in}(s) = \dfrac{100}{s^2+4}$ in Eq. (1.3)

$$V_{o3}(s) = \frac{100s}{2(s+10)(s^2+4)}$$

$$V_{o3}(s) = \frac{50s}{(s+10)(s^2+4)}$$

Inverse Laplace

$$v_o(t) = \frac{125}{26} \cos 2t + \frac{25}{26} \sin 2t - \frac{125}{26} e^{-10t} \tag{1.9}$$

Add Eq. (1.7), Eq. (1.8), Eq. (1.9)

$$v_o(t) = \frac{15}{2} e^{-10t} + 25te^{-10t} - \frac{5}{2} e^{-10t} + \frac{125}{26} \cos 2t + \frac{25}{26} \sin 2t - \frac{125}{26} e^{-10t}$$

(c) Us your answer to b(i) to find the time constant of the circuit?

We know that the general solution to impulse response is

$$v(t) = v_o + v_a e^{-\frac{t}{\tau}}$$

Compairing it with Eq.1.6 $\dfrac{1}{\tau} = 10$ which means time constant is $\tau = \dfrac{1}{10} = 0.1 sec$

(d) Using your answer to (a) and b(ii), verify the final value theorem for a unit step input.

Final value property determines the final value of the waveform $v_o(t)$ from the value of $\lim_{s \to 0} sV_o(s)$.Applying final value theorem on Eq. (1.3)

$$\lim_{t \to \infty} v_o(t) = \lim_{s \to 0} sV_o(s) = \frac{s^2}{2s+20} = 0$$

Which means our system will converge to zero in time domain as time goes to infinity.Now applying final value theorem to Eq.1.7

$$\lim_{t \to \infty} v_o(t) = \lim_{s \to 0} sV_o(s) = \frac{s^2}{s(2s+20)} = 0$$

(e) You know that transfer function is evaluated at zero initial conditions. So there seems to be no way of incorporating non-zero initial conditions to the transfer function method. However, giving an step response at the input is equivalent to giving the system an initial condition. Now using the initial value theorem, show that the case b(ii) is equivalent to solving the circuit with $v_o(0) = 0.5$ V and $v_i(t) = 0$ for $t > 0$.

Applying the initial value theorem on Eq. (1.7)

$$\lim_{t \to 0} v_o(t) = \lim_{s \to \infty} sV_o(s) = \frac{s^2}{s(2s+20)}$$

$$\lim_{s \to \infty} \frac{1}{2 + \frac{20}{s}} = \frac{1}{2}$$

(f) Now find $v_o(t)$ if a DC input voltage of 1 V is applied at $t = 0$ and $v_o(0) = 0.5$ V.

Transfer Function of the system is

$$G(s) = \frac{s}{(2s+20)}$$

$$\frac{V_o(s)}{V_{in}(s)} = \frac{s}{(2s+20)}$$

$$V_o(s) = \frac{s}{2(s+10)} V_{in}(s)$$

$$V_o(s)(s+10) = s\frac{V_{in}(s)}{2}$$

$$V_o(s)s + 10V_o(s) = s\frac{V_{in}(s)}{2}$$

Inverse Laplace :

$$\dot{v_o} + 10v_o = \frac{\dot{v_{in}}}{2}$$

Laplace transform of above equation is

$$V_o(s)s - V_o(0) + 10V_o(s) = V_{in}(s)\frac{s}{2} - \frac{V_{in}(0)}{2}$$

$$V_o(s)s + 10V_o(s) = V_{in}(s)\frac{s}{2} - \frac{V_{in}(0)}{2} + V_o(0)$$

$$V_o(s)(s+10) = V_{in}(s)\frac{s}{2} - \frac{V_{in}(0)}{2} + V_o(0)$$

$$V_o(s) = V_{in}(s)\frac{s}{2(s+10)} - \frac{V_{in}(0)}{2(s+10)} + \frac{1}{s+10}V_o(0) \tag{1.10}$$

Put $V_{in}(s) = 0$, $V_{in}(0) = 0$, $V_o(0) = 0.5$ in Eq. (1.10)

$$V_o(s) = \frac{s}{2(s+10)}$$

$$v_{o1} = \frac{1}{2}e^{-10t} \tag{1.11}$$

Put $V_{in}(s) = \frac{1}{s}$, $V_{in}(0) = 0$, $V_o(0) = 0$ in Eq. (1.10)

$$V_o(s) = \frac{s}{2s(s+10)}$$

$$v_{o2} = \frac{1}{2}e^{-10t} \tag{1.12}$$

Adding Eq. (1.11) and Eq. (1.12)

$$v_o = \frac{1}{2}e^{-10t} + \frac{1}{2}e^{-10t}$$

Problem 2

Two carts of masses m_1 and m_2 with negligible rolling friction are connected to two springs (spring constant k_1 and k_2) and dampers (damping constant b_1 and b_2) as shown in the figure. An input force $u(t)$ is applied on m_1 and the output is the position of m_2. The displacements of m_1 and m_2 from their equilibrium positions are x_1 and x_2 respectively.

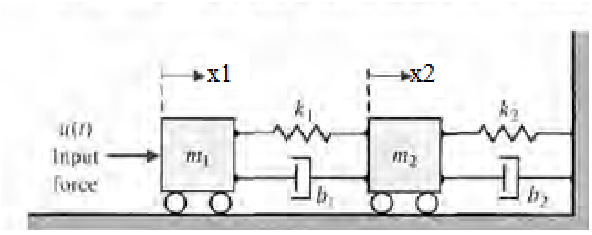

Figure 4: Double mass-spring system

(a) Write down the differential equation for the displacement x_1 of mass m_1.

For writing the differential equation for m_1 apply the super position principle, m_1 moving m_2 stationary

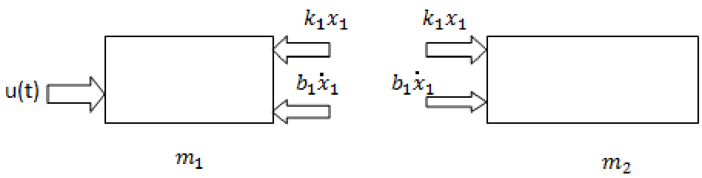

Figure 5: Free Body Diagram of Spring-Mass-Damper System m_1 moving m_2 stationary

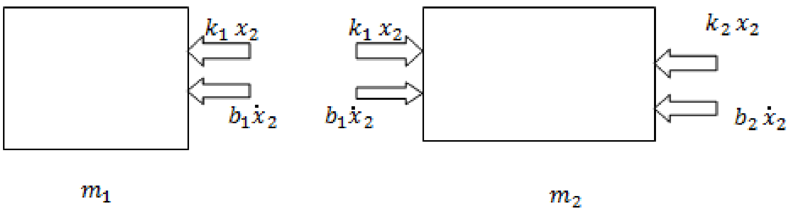

Figure 6: Free Body Diagram of Spring-Mass-Damper System m_2 moving m_1 stationary

Using free body diagram, equation of motion form_1 according to Newton's second law is

$$F_{net} = m_1 a = u(t) - (x_1 - x_2)k_1 - (\dot{x_1} - \dot{x_2})b_1$$

$$u(t) = m_1\ddot{x_1} + (\dot{x_1} - \dot{x_2})b_1 + (x_1 - x_2)k_1 \tag{2.1}$$

(b) Write down the differential equation for displacement x_2 of mass m_2.

Using free body diagram, equation of motion fo rm_2 according to Newton's second law is

$$F_{net} = m_2 a = -k_2 x_2 - b_2 \dot{x_2} + (x_1 - x_2)k_1 + (\dot{x_1} - \dot{x_2})b_1$$

$$m_2 \ddot{x_2} - (\dot{x_1} - \dot{x_2})b_1 - (x_1 - x_2)k_1 + k_2 x_2 + b_2 \dot{x_2} = 0 \tag{2.2}$$

(c) Take the Laplace transform of equations in (a) and (b).

Laplace transform of Eq. (2.1)

$$U(s) = m_1 s^2 X_1(s) + k_1 X_1(s) + b_1 s X_1(s) - k_1 X_2(s) - b_1 s X_2(s)$$

$$U(s) = (m_1 s^2 + k_1 + b_1 s)X_1(s) - (k_1 + b_1 s)X_2(s)$$

Laplace transform of Eq.2.2

$$0 = m_2 s^2 X_2(s) - k_1 X_1(s) - b_1 s X_1(s) + k_1 X_2(s) + b_1 s X_2(s) + b_2 s X_2(s) + k_2 X_2(s)$$

$$0 = -(k_1 + b_1 s)X_1(s) + (m_2 s^2 + k_1 + b_1 s + b_2 s + k_2)X_2(s)$$

(d) Find the transfer function $\frac{X_2(s)}{U(s)}$. [Hint: Cramer's rule]

$$X_2(s) = \frac{\begin{vmatrix} m_1 s^2 + k_1 + b_1 s & U(s) \\ -(k_1 + b_1 s) & 0 \end{vmatrix}}{\begin{vmatrix} m_1 s^2 + k_1 + b_1 s & -(k_1 + b_1 s) \\ -(k_1 + b_1 s) & m_2 s^2 + k_1 + b_1 s + b_2 s + k_2 \end{vmatrix}}$$

$$X_2(s) = \frac{-(k_1 + b_1 s)U(s)}{(m_1 s^2 + k_1 + b_1 s)(m_2 s^2 + k_1 + b_1 s + b_2 s + k_2) - (k_1 + b_1 s)^2}$$

$$\frac{X_2(s)}{U(s)} = \frac{(k_1 + b_1 s)}{(m_1 s^2 + k_1 + b_1 s)(m_2 s^2 + k_1 + b_1 s + b_2 s + k_2) - (k_1 + b_1 s)^2} \tag{2.3}$$

(e) Find the approximate transfer function if $m_2 >> m_1$, $k_2 >> k_1$ and $b_2 >> b_1$.

Rearranging Eq. (2.3)

$$\frac{X_2(s)}{U(s)} = \frac{(k_1 + b_1 s)}{(m_1 m_2)s^4 + (m_2 b_1 + m_1 b_1 + m_1 b_2)s^3 + (m_2 k_1 + m_1 k_2 + m_1 k_1 + b_2 b_1)s^2 + (b_2 k_1 + b_1 k_2)s + k_1 k_2}$$

Taking $\frac{1}{{m_2}^2 {k_2}^2 {b_2}^2}$ common from above equation approximate transfer function of the system will be

$$\frac{X_2(s)}{U(s)} = \frac{(k_1 + b_1 s)}{(m_2 b_1 + m_1 b_2)s^3 + (m_2 k_1 + m_1 k_2)s^2 + (b_2 k_1 + b_1 k_2)s}$$

Problem 3

A common actuator based on the laws of motors and generators and used in control systems is the direct current (DC) motor to provide rotary motion. The electric equivalent circuit of the armature and the free-body diagram of the rotor are shown in the following figure.

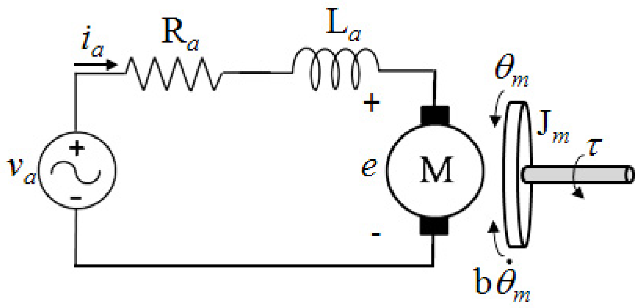

Figure 7: A simple model of a DC Motor. Quantities i_a, R_a and L_a are armature current, resistance and inductance respectively. Quantities J_m and θ_m are moment of inertia and rotational displacement of the rotor while τ and $b\dot{\theta}_m$ are torque and frictional torque acting on it.

The motor equations give the torque τ on the rotor in terms of the armature current i_a as

$$\tau = K_t i_a$$

and express the back emf in terms of the shaft's rotational velocity $\dot{\theta}_m$ as

$$e = K_e \dot{\theta}_m = K_e \omega_m$$

(a) Write down the differential equations of this electromechanical system.

Applying KVL to the circuit,

$$v_a = i_a R_a + L_a \frac{di_a}{dt} + e$$

$$i_a R_a + L_a \frac{di_a}{dt} + K_e \omega_m = v_a \tag{3.1}$$

Applying Newton's second law to the mechanical part,

$$J_m \dot{\omega}_m = \tau - b\omega_m$$

$$\implies J_m \dot{\omega}_m + b\omega_m = \tau$$

$$J_m \dot{\omega}_m + b\omega_m = K_t i_a. \tag{3.2}$$

(b) Derive $\dfrac{\Omega_m(s)}{V_a(s)}$, the transfer function between rotational velocity and armature voltage.

Taking Laplace transform of Eq. (3.1),

$$R_a I_a(s) + L_a s I_a(s) + K_e \Omega_m(s) = V_a(s)$$

$$(R_a + L_a s) I_a(s) + K_e \Omega_m(s) = V_a(s)$$

$$I_a(s) = \frac{1}{L_a s + R_a} V_a(s) - \frac{K_e}{L_a s + R_a} \Omega_m(s) \tag{3.3}$$

Now taking Laplace transform of Eq. (3.2),

$$J_m s\Omega_m(s) + b\Omega_m(s) = K_t I_a(s)$$

$$I_a(s) = \frac{J_m s + b}{K_t}\Omega_m(s) \tag{3.4}$$

Now equating Eqs. (3.3) and (3.4),

$$\frac{1}{L_a s + R_a}V_a(s) - \frac{K_e}{L_a s + R_a}\Omega_m(s) = \frac{J_m s + b}{K_t}\Omega_m(s)$$

$$V_a(s) = \left(\frac{(R_a + L_a s)(J_m s + b)}{K_t} + K_e\right)\Omega_m(s)$$

$$\implies \frac{\Omega_m(s)}{V_a(s)} = \frac{K_t}{(J_m s + b)(L_a s + R_a) + K_e K_t} \tag{3.5}$$

(c) Using part (b), derive $\frac{\Theta_m(s)}{V_a(s)}$, the transfer function between rotational displacement and armature voltage.

Now, because $\omega_m = \dot{\theta}_m$, in terms of their Laplace transforms, we can write

$$\Omega_m(s) = s\Theta_m(s)$$

Now using Eq. (3.5),

$$\frac{s\Theta_m(s)}{V_a(s)} = \frac{K_t}{(J_m s + b)(L_a s + R_a) + K_e K_t}$$

$$\implies \frac{\Theta_m(s)}{V_a(s)} = \frac{K_t}{s\left((J_m s + b)(L_a s + R_a) + K_e K_t\right)}$$

(d) Given that at $t = 0$, a constant armature voltage V_o is applied. Using part (b), derive an expression for the rotational velocity of the shaft in steady state.

As, v_a is equal to a constant voltage V_o. Therefore,

$$V_a(s) = \frac{V_o}{s}$$

Now, from Eq. (3.5)

$$\Omega_m(s) = \left(\frac{K_t}{(J_m s + b)(L_a s + R_a) + K_e K_t}\right)\frac{V_o}{s} = \frac{K_t V_o}{s((J_m s + b)(L_a s + R_a) + K_e K_t)}$$

For steady state value,

$$\omega_{m(ss)} = \lim_{t\to\infty}\omega_m = \lim_{s\to 0} s\Omega(s) = \lim_{s\to 0} s\frac{K_t V_o}{s\left((J_m s + b)(L_a s + R_a) + K_e K_t\right)}$$

$$\implies \omega_{m(ss)} = \frac{K_t V_o}{R_a b + K_e K_t} \tag{3.6}$$

As constant voltage is applied, rotational velocity of motor will have some transient response in start but eventually achieve a constant value.

(e) What is the steady state value of rotational velocity if friction in the motor is very large?Think about what would you expect in such a case and whether your answer makes sense or not.

As $b \to \infty$, the denominator of Eq. (3.6) $R_a b + K_e K_t \to \infty$ Therefore,

$$\omega_{m(ss)} = \frac{K_t V_o}{R_a b + K_e K_t} \to 0$$

The above expression shows that even a constant voltage is applied, the rotational velocity of motor eventually gets very low as resistance/friction in the motor gets very large. This means that effectively the motor does not move at all when the friction in the motor is very large.

(f) What is the steady state value of rotational velocity if both armature resistance and friction in the motor are negligible ($R_a \to 0$, $b \to 0$), and back emf is small ($K_e \to 0$)? Think about what would you expect in such a case and whether your answer makes sense or not.

As $R_a \to 0$, $b \to 0$, and $K_e \to 0$ the expression $R_a b + K_e K_t \to 0$. Therefore,

$$\omega_{m(ss)} = \frac{K_t V_o}{R_a b + K_e K_t} \to \infty$$

The above expression shows that even if a constant voltage is applied, the rotational velocity of the motor keeps on getting larger and larger when resistance/friction in the motor is very low (almost zero).

(g) Now suppose that a load is attached to the motor through a gear system with gear ratio $N = \frac{N_l}{N_m}$, where N_l and N_m are number of teeth of gears at the load's input and motor's output respectively. Find the steady-state rotational velocity of the rotor if the load's moment of inertia is J_l and its rotation experiences a friction with damping constant b_l.

Using Eq. (3.6)

$$\omega_{m(ss)} = \frac{K_t V_o}{R_a b + K_e K_t}$$

Now due to attached load through gear system

$$b = b_m + \left(\frac{N_m}{N_l}\right)^2 b_l$$

so

$$\omega'_{m(ss)} = \frac{K_t V_o}{R_a b_m + R_a \left(\frac{N_m}{N_l}\right)^2 b_l + K_e K_t}$$

(h) Compare your answers to (d) and (g) if $N_l > N_m$.

By rearranging the answer to (g)

$$\omega'_{m(ss)} = \frac{K_t V_o N_l^2}{N_l^2 R_a b_m + R_a N_m^2 b_l + N_l^2 K_e K_t}$$

If $N_l > N_m$ then $\omega'_{m(ss)} > \omega_{m(ss)}$