EE361: Control Systems

Homework 1

Due Fri Feb 22, 2 PM *Spring 2019*

Some tips to avoid a plagiarism case:

- Do not copy the solutions of your classmates.
- Your are encouraged to discuss the problems with your classmates in whatever way you like but make sure to REPRODUCE YOUR OWN SOLUTIONS in what you submit for grading.
- Cite all the online sources that you get help from.
- Keep your work in a secure place.

Problem 1

Consider an RC circuit shown in figure below.

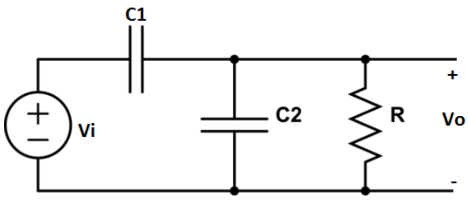

Figure 1: An RC Circuit

(a) Find the transfer function $\dfrac{V_o(s)}{V_i(s)}$ of the electric circuit using the impedance model.

[Hint: Perform nodal analysis with admittances to make your life easier in this case.]

(b) Using your answer to (a), find the output voltage $v_o(t)$ if $R = 100$ kΩ, $C_1 = 500$ nF and $C_2 = 500$ nF for each of the following cases, where the input voltage v_i, measured in volts, is applied at $t = 0$.

(i) $v_i(t) = \delta(t)$
(ii) $v_i(t) = 1$
(iii) $v_i(t) = e^{-10t}$
(iv) $v_i(t) = \sin 2t$
(v) $v_i(t) = 15 - 5e^{-10t} + 50 \sin 2t$

[Hint for (v): Apply linearity to the 'particular solutions' in (ii), (iii) and (iv).]

(c) Us your answer to b(i) to find the time constant of the circuit?

(d) Using your answer to (a) and b(ii), verify the final value theorem for a unit step input.

(e) You know that transfer function is evaluated at zero initial conditions. So there seems to be no way of incorporating non-zero initial conditions to the transfer function method. However, in this particular circuit, giving a step input is equivalent to giving the circuit an initial condition. Now using the initial value theorem, show that the solution to b(ii) is equivalent to solving the circuit with $v_o(0) = 0.5$ V but $v_i(t) = 0$ for $t > 0$.

(f) Now find $v_o(t)$ if a DC input voltage of 1 V is applied at $t = 0$ and $v_o(0) = 0.5$ V.

Problem 2

Two carts of masses m_1 and m_2 with negligible rolling friction are connected to two springs (spring constant k_1 and k_2) and dampers (damping constant b_1 and b_2) as shown in the figure. An input force $u(t)$ is applied on m_1 and the output is the position of m_2. The displacements of m_1 and m_2 from their equilibrium positions are x_1 and x_2 respectively.

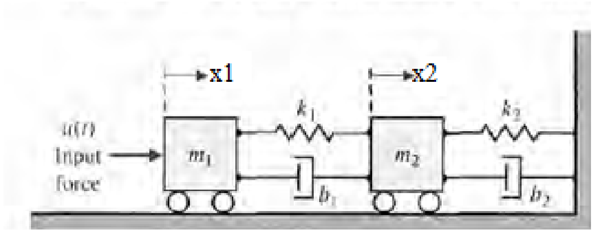

Figure 2: Double mass-spring system

(a) Write down the differential equation for the displacement x_1 of mass m_1.

(b) Write down the differential equation for displacement x_2 of mass m_2.

(c) Take the Laplace transform of equations in (a) and (b).

(d) Find the transfer function $\dfrac{X_2(s)}{U(s)}$. [Hint: Cramer's rule]

(e) Find the approximate transfer function if $m_2 >> m_1$, $k_2 >> k_1$ and $b_2 >> b_1$.

Problem 3

A common actuator based on the laws of motors and generators used in control systems is the direct current (DC) motor to provide rotary motion. The electric equivalent circuit of the armature and the free-body diagram of the rotor are shown in the following figure.

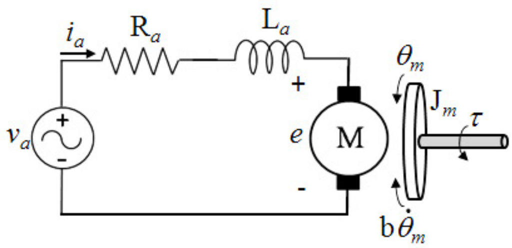

Figure 3: A simple model of a DC Motor. Quantities i_a, R_a and L_a are armature current, resistance and inductance respectively. Quantities J_m and θ_m are moment of inertia and rotational displacement of the rotor while τ and $b\dot{\theta}_m$ are torque and frictional torque acting on it.

The motor equations give the torque τ on the rotor in terms of the armature current i_a as

$$\tau = k_t i_a,$$

and express the back emf in terms of the shaft's rotational velocity $\dot{\theta}_m$ as

$$e = k_e \dot{\theta}_m.$$

(a) Write down the differential equations of this electromechanical system.

(b) Using part (a), derive $\frac{\Omega_m(s)}{V_a(s)}$, the transfer function between rotational velocity of rotor and the armature voltage.

(c) Using part (b), derive $\frac{\Theta_m(s)}{V_a(s)}$, the transfer function between rotational displacement of rotor and the armature voltage.

Given that at $t = 0$, a constant armature voltage v_i is applied.

(d) Using part (c), derive an expression for the rotational velocity of the rotor in steady state (as $t \to \infty$) [Hint: Use the final value theorem.]

(e) What is the steady-state value of the rotational velocity if friction in the motor is very large? Think about what would you expect in such a case and whether your answer makes sense or not.

(f) What is the steady-state value of the rotational velocity if both armature resistance and friction in the motor are negligible ($R_a \to 0$, $b \to 0$), and back emf is small ($k_e \to 0$)? Think about what would you expect in such a case and whether your answer makes sense or not.

(g) Now suppose that a load is attached to the motor through a gear system with gear ratio $N = \frac{N_l}{N_m}$, where N_l and N_m are number of teeth of gears at the load's input and motor's output respectively. Find the steady-state rotational velocity of the rotor if the load's moment of inertia is J_l and its rotation experiences a friction with damping constant b_l.

(h) Compare your answers to (d) and (g) if $N_l > N_m$.